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ON MHV FORM FACTORS IN SUPERSPACE FOR $\mathcal{N}=4$ SYM THEORY

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Abstract

In this paper we develop a supersymmetric version of a unitarity cut method for form factors of operators from the $\mathcal{N}=4$ stress-tensor current supermultiplet T^{AB}. The relation between the super form factor with super momentum equal to zero and the logarithmic derivative of the superamplitude with respect to the coupling constant is discussed and verified at the tree- and one-loop level for any MHV n-point ($n \geq 4$) super form factor. The explicit $\mathcal{N}=4$ covariant expressions for n-point MHV tree- and one-loop form factors are obtained. As well, the ansatz for the two-loop three-point MHV super form factor is suggested in the planar limit, based on the reduction procedure for the scalar integrals suggested in our previous work. The different soft and collinear limits in the MHV sector at the tree- and one-loop level are discussed.

Keywords: Super Yang-Mills Theory, form factors, superspace.

Contents

1 Introduction

Much attention in the past few years has been paid to the study of the planar limit for the scattering amplitudes in the $\mathcal{N} = 4$ SYM theory. It is believed that the hidden symmetries responsible for integrability properties of this theory completely fix the structure of the amplitudes (the S-matrix of the theory). The hints that the S-matrix for the $\mathcal{N} = 4$ SYM theory can be fixed by some underlying integrable structure were found at weak [1, 2, 3] and strong [3, 4] coupling regimes.

The progress in understanding of the S-matrix structure of the $\mathcal{N} = 4$ SYM theory has been made mainly due to achievements in the development of the computational methods beyond the ordinary textbook Feynman rules, such as MHV vertex expansion [5], recursion relations [6], unitarity cut constructibility techniques [7], generalized unitarity [8] and their $\mathcal{N} = 4$ covariant generalizations [9, 10].

There is another class of objects of interest in the $\mathcal{N}=4$ SYM theory which resemble the amplitudes – the form factors which are the matrix elements of the form

$$\langle 0|\mathcal{O}|p_1^{\lambda_1},\dots,p_n^{\lambda_n}\rangle, \tag{1.1}$$

where $\mathcal{O}$ is some gauge invariant operator which acts on the vacuum and produces some state $|p_1^{\lambda_1},\dots,p_n^{\lambda_n}\rangle$ with momenta $p_1,\dots,p_n$ and helicities $\lambda_1,\dots,\lambda_n$[1]. The S-matrix operator is assumed in both cases.

The two-point form factor was studied long time ago in [11] and later, using the $\mathcal{N}=3$ superfield formalism, the tree-level form factors were derived in [12]. Recently, the strong coupling limit of form factors has been studied in [13] and the weak coupling regime in [14, 15]. The motivations for the systematic study of these objects are

- it might help in understanding of the symmetry properties of the amplitudes [1, 2]. It is believed that the symmetries completely fix the amplitudes of the $\mathcal{N}=4$ SYM theory and it is interesting to see whether they fix/restrict the form factors as well;
- the form factors are the intermediate objects between the fully on-shell quantities such as the amplitudes and the fully off-shell quantities such as the correlation functions (which are one of the central objects in AdS/CFT). Since the powerful computational methods have appeared recently for the amplitudes in $\mathcal{N}=4$ SYM [9, 10], it would be desirable to have some analog of them for the correlation functions [16]. The understanding of the structure of form factors and the development of computational methods might shed light on the correlation functions;
- recently, it was observed [17] that in the limit when the distances between the operators in the correlation functions become light-like, there is a simple relation between the $n+l$-point correlation function with l Lagrangian insertions and the integrand of the MHV n-point amplitude at the l loop level. The form factors, as was explained above, are the intermediate objects between completely on-shell/off-shell quantities, so they might help in understanding the relations discussed in [17];
- also, it might be useful for understanding of the relation between the conventional description of the gauge theory in terms of local operators and its (possible) description in terms of Wilson loops. The latter fact is the so-called amplitude/Wilson loop duality which originated in [18, 19, 20]. This duality was intensively studied in the weak and strong coupling regimes and tested in different cases, and its generalizations to the non-MHV amplitudes were proposed in [21]. Recently, a similar relation between certain form factors and Wilson loops was observed at the one-loop level in the weak coupling regime [14] which should be verified at a higher loop level.

To make progress in the above-mentioned directions, the perturbative computations at several first orders of perturbative theory (PT) are likely required. For this purpose,

[1]Note that scattering amplitudes in "all ingoing" notation can schematically be written as $\langle 0|p_1^{\lambda_1},\dots,p_n^{\lambda_n}\rangle$.

proper computational tools beyond the ordinary textbook Feynman rules which are bulky at higher order of PT and/or at a large number of external particles are desirable. The recent attempts at systematic study of form factors using the $\mathcal{N}=1$ superfield formalism has been carried out in [15] and the component version of recursion relations and unitarity cut constructibility techniques have been applied in [14]. The latter method is likely more efficient due to the experience with the amplitude computations but still there is a lack of full $\mathcal{N}=4$ supersymmetry covariance, which makes the summation over the intermediate states at a higher loop level ($l \geq 2$) rather cumbersome [22].

The aim of this paper is to discuss the $\mathcal{N}=4$ generalization of the unitarity cut constructibility techniques for the form factors of the $\mathcal{N}=4$ stress tensor current supermultiplet T^{AB} (more accurately, its chiral truncation). We introduce the "superstate super form factor" at the tree level and explicitly define its MHV part. Then we re-derive the one-loop correction to its MHV part for an arbitrary number of legs in external state $|p_1^{\lambda_1},\ldots,p_n^{\lambda_n}\rangle$ in the covariant $\mathcal{N}=4$ notation. We also suggest an ansatz for the 3-point MHV super form factor at two loops based on the conjecture that the basis of scalar integrals for form factors can be obtained from that of dual pseudoconformal integrals arising in calculations of the amplitudes via the reduction procedure suggested in [15]. At the end, we also discuss various soft and collinear limits at the tree- and one-loop level.

2 Form factors of the $\mathcal{N}=4$ stress tensor supermultiplet and half-BPS operators in superspace

2.1 Preliminaries and notation

It is convenient to describe the pure on-shell scattering amplitudes using the so-called $\mathcal{N}=4$ on-shell momentum superspace [23]. This superspace is parameterized in terms of $SL(2,C)$ spinors $\lambda_\alpha, \tilde{\lambda}_{\dot\alpha}, \alpha, \dot\alpha = 1,2$ and Grassmannian coordinates $\eta^A, A=1,\ldots,4$ which are Lorentz scalars and $SU(4)_R$ vectors

$$\text{On-shell } \mathcal{N}=4 \text{ momentum superspace} = \{\lambda_\alpha, \tilde{\lambda}_{\dot\alpha}, \eta^A\}. \tag{2.2}$$

Note that this superspace is chiral (i.e. it can be parameterized only in terms of Grassmannian variables η, or, equivalently, in terms of their conjugated partners $\bar\eta$). The generators of supersymmetry algebra relevant for our discussion, in the case when we want to describe the n-particle amplitude, are realized in on-shell momentum superspace as

$$\begin{aligned} 4 \text{ translations } p_{\alpha\dot\alpha} &= \sum_{i=1}^{n} \lambda_\alpha^i \tilde{\lambda}_{\dot\alpha}^i, \\ 8 \text{ supercharges } q_\alpha^A &= \sum_{i=1}^{n} \lambda_\alpha^i \eta_i^A, \end{aligned}$$

$$8 \text{ conjugated supercharges } \bar{q}_{A\dot{\alpha}} = \sum_{i=1}^{n} \tilde{\lambda}^i_{\dot{\alpha}} \frac{\partial}{\partial \eta^A_i} = \sum_{i=1}^{n} \tilde{\lambda}^i_{\dot{\alpha}} \partial_{iA}, \tag{2.3}$$

where λ^i_α and $\tilde{\lambda}^i_{\dot{\alpha}}$ correspond to the i-th particle with on-shell momentum $p^i_\mu(\sigma^\mu)_{\alpha\dot{\alpha}} = \lambda^i_\alpha \tilde{\lambda}^i_{\dot{\alpha}}$, $p_i^2 = 0$, $(\sigma^\mu)_{\alpha\dot{\alpha}}$ is the Pauli sigma matrix, μ is the index of the Lorentz group vector representation. In the on-shell momentum superspace the creation/annihilation operators

$$\{g^-,\ \Gamma^A,\ \phi^{AB},\ \bar{\Gamma}^A,\ g^+\},$$

of the $\mathcal{N}=4$ supermultiplet, for the on-shell states which are two physical polarizations of gluons $|g^-\rangle, |g^+\rangle$, four fermions $|\Gamma^A\rangle$ with positive and four fermions $|\bar{\Gamma}^A\rangle$ with negative helicity, and three complex scalars $|\phi^{AB}\rangle$ (anti-symmetric in $SU(4)_R$ indices AB) can be combined together into one $\mathcal{N}=4$ invariant superstate ("superwave-function") $|\Omega_i\rangle$

$$|\Omega_i\rangle = \left(g_i^+ + \eta^A \Gamma_{i,A} + \frac{1}{2!}\eta^A\eta^B\phi_{i,AB} + \frac{1}{3!}\eta^A\eta^B\eta^C\varepsilon_{ABCD}\bar{\Gamma}^D_i + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\varepsilon_{ABCD}g_i^- \right)|0\rangle, \tag{2.4}$$

here i corresponds to the on-shell momentum $p^i_{\alpha\dot{\alpha}} = \lambda^i_\alpha \tilde{\lambda}^i_{\dot{\alpha}}$, $p_i^2 = 0$ carried by the on-shell particle. Then one can write the n-point superamplitude as

$$\mathcal{A}_n(\lambda, \tilde{\lambda}, \eta) = \langle 0| \prod_{i=1}^{n} \Omega_i |0\rangle, \tag{2.5}$$

where the average $\langle 0| \dots |0\rangle$ is understood with respect to some particular (for example component) formulation of the $\mathcal{N}=4$ SYM theory. The "all ingoing" notation as well as the color decomposition, the color ordering and the planar limit[2] are implemented through out this paper for all objects containing $|\Omega\rangle$ superstate.

We also use the standard $\mathcal{N}=4$ coordinate superspace, which is convenient for the description of supermultiplets of fields or operators and which is parameterized by the following coordinates:

$$\mathcal{N}=4 \text{ coordinate superspace} = \{x^{\alpha\dot{\alpha}},\ \theta^A_\alpha,\ \bar{\theta}_{A\dot{\alpha}}\}, \tag{2.6}$$

where $x_{\alpha\dot{\alpha}}$ are bosonic coordinates and θ's, which are $SU(4)_R$ vectors and Lorentz $SL(2,C)$ spinors, are fermionic ones. The generators of supersymmetry algebra relevant for our discussion are realized in this superspace as

$$\begin{aligned}
4 \text{ translations } P_{\alpha\dot{\alpha}} &= q_{\alpha\dot{\alpha}}, \\
8 \text{ supercharges } Q^A_\alpha &= \frac{\partial}{\partial \theta^\alpha_A} - \bar{\theta}^{\dot{\alpha}A} q_{\alpha\dot{\alpha}}, \\
8 \text{ conjugated supercharges } \bar{Q}_{A\dot{\alpha}} &= -\frac{\partial}{\partial \bar{\theta}^{A\dot{\alpha}}} + \theta^\alpha_A q_{\alpha\dot{\alpha}},
\end{aligned} \tag{2.7}$$

[2] $g \to 0$ and $N_c \to \infty$ of $SU(N_c)$ gauge group so that $\lambda = g^2 N_c$ =fixed.

where one considers the Fourier transformed generators for bosonic coordinates $x^{\alpha\dot{\alpha}} \to q_{\alpha\dot{\alpha}}$.

The full $\mathcal{N}=4$ coordinate superspace is obviously non-chiral in contrast to the on-shell momentum superspace. The $\mathcal{N}=4$ supermultiplet of fields (containing ϕ^{AB} scalars, $\psi^A_\alpha, \bar{\psi}^A_{\dot{\alpha}}$ fermions and $F^{\mu\nu}$– the gauge field strength tensor, all in the adjoint representation of $SU(N_c)$ gauge group) is realized in $\mathcal{N}=4$ coordinate superspace as a constrained superfield $W^{AB}(x,\theta,\bar{\theta})$ with the lowest component $W^{AB}(x,0,0)=\phi^{AB}(x)$. W^{AB} in general is not a chiral object and satisfies several constraints: a self-duality constraint

$$W^{AB}(x,\theta,\bar{\theta}) = \overline{W_{AB}}(x,\theta,\bar{\theta}) = \frac{1}{2}\epsilon^{ABCD}W_{CD}(x,\theta,\bar{\theta}), \tag{2.8}$$

which implies $\phi^{AB}=\overline{\phi_{AB}}=\frac{1}{2}\epsilon^{ABCD}\phi_{CD}$ and two additional constraints[3]

$$\begin{aligned} &D^\alpha_C W^{AB}(x,\theta,\bar{\theta}) = -\frac{2}{3}\delta^{[A}_C D^\alpha_L W^{B]L}(x,\theta,\bar{\theta}),\\ &\bar{D}^{\dot{\alpha}(C}W^{A)B}(x,\theta,\bar{\theta}) = 0, \end{aligned} \tag{2.9}$$

where D^A_α is a standard coordinate superspace derivative[4]. Note that in this formulation the full $\mathcal{N}=4$ supermultiplet of fields is on-shell in the sense the algebra (more precisely the last two anti commutators) of the generators $Q^A_\alpha, \bar{Q}_{B\dot{\alpha}}$ for the supersymmetric transformation of the fields in this supermultiplet

$$\{Q^A_\alpha, \bar{Q}_{B\dot{\alpha}}\} = 2\delta^A_B P_{\alpha\dot{\alpha}},\ \{Q^A_\alpha, Q^B_\beta\}=0,\ \{\bar{Q}_{A\dot{\alpha}}, \bar{Q}_{B\dot{\beta}}\}=0 \tag{2.10}$$

is closed only if the fields obey their equations of motion (in addition the closure of the algebra requires the compensating gauge transformation [24]).

Using (2.9) one can write [24] any W^{AB} for a particular choice of A,B in the form that is independent of half of θ's and $\bar{\theta}$'s. This property is invariant under the transformations of θ's and $\bar{\theta}$'s with respect to some $SU(4)_R$ subgroup, while for all other values of A and B W^{AB} contains the full dependence on θ's and $\bar{\theta}$'s. More accurately, one breaks $SU(4)_R$ group into two $SU(2)$ and $U(1)$

$$SU(4)_R \ \to\ SU(2)\times SU(2)'\times U(1), \tag{2.11}$$

so that the index A of R-symmetry group $SU(4)_R$ splits into

$$A \ \to\ (+a|-a'), \tag{2.12}$$

where $+a$ and $-a'$ corresponds to two copies of $SU(2)$ and $\pm$ corresponds to the $U(1)$ charge (we do not write the $U(1)$ factor explicitly hereafter, and use the notation $+a\equiv a$

[3] $[*,\star]$ denotes antisymmetrization in indices, while $(*,\star)$ denotes symmetrization in indices.

[4] which is $D^A_\alpha = \partial/\partial\theta^\alpha_A + i\bar{\theta}^{A\dot{\alpha}}\partial/\partial x^{\alpha\dot{\alpha}}$.

and $-a' \equiv \dot{a}$); and then takes particular (ab) projection of W^{AB} after which the constraints (2.9) become [24]

$$\begin{aligned} D^{\dot{c}}_{\alpha} W^{ab}(x,\theta,\bar{\theta}) &= 0, \\ \bar{D}_{c\dot{\alpha}} W^{ab}(x,\theta,\bar{\theta}) &= 0. \end{aligned} \tag{2.13}$$

The latter constraints are called the Grassmannian analyticity conditions (see [24] and references therein for details) resolving which one gets

$$W^{ab}(x,\theta,\bar{\theta}) = W^{ab}(X,\theta^{c},\bar{\theta}_{\dot{c}}), \tag{2.14}$$

where X is a chiral coordinate

$$X^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} + i(\theta^{a}\bar{\theta}_{a})^{\alpha\dot{\alpha}} - i(\theta^{\dot{a}}\bar{\theta}_{\dot{a}})^{\alpha\dot{\alpha}}. \tag{2.15}$$

The full $\mathcal{N}=4$ supermultiplet can be truncated to the so-called $\mathcal{N}=4$ vector chiral multiplet $\mathcal{W}^{AB}$, which is closed under the chiral part of supersymmetry generators Q^{A}_{α} and contains the self-dual part of the $\mathcal{N}=4$ supermultiplet: ϕ^{AB} scalars, ψ^{A}_{α} fermions and $F^{\alpha\beta}$ that is a self-dual part of $F^{\mu\nu}$ defined as

$$F^{\mu\nu} \to F^{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon^{\alpha\beta}\tilde{F}^{\dot{\alpha}\dot{\beta}} + \epsilon^{\dot{\alpha}\dot{\beta}}F^{\alpha\beta}.$$

$\mathcal{W}^{AB}$ can be obtained by setting $\bar{\theta}=0$ in W^{AB}. For the $\mathcal{W}^{ab}$ component (projection) of $\mathcal{W}^{AB}$ one gets the following expansion in the Grassmannian coordinates [24]:

$$\mathcal{W}^{ab}(x,\theta^{c}) = W^{ab}(x,\theta^{c},0) = \phi^{ab} - \mathrm{i}\sqrt{2}(\theta^{[a}\psi^{b]}) + \frac{\mathrm{i}}{\sqrt{2}}(\theta^{[a}\theta^{b]}F) + O(g). \tag{2.16}$$

Here $O(g)$ corresponds to the terms proportional to commutators of $SU(N_c)$ matrices in the fundamental representation and $(\ldots)$ stands for the contractions of $SL(2,C)$ indices. Note that in the latter expression one can write ϕ^{ab} as $\phi_{\dot{a}\dot{b}}$ due to the self-duality constraint. We also want to point out that the component fields in $\mathcal{W}^{AB}$ are off-shell [24], i.e. the component fields in $\mathcal{W}^{AB}$ are arbitrary and the chiral part of the algebra $\{Q^{A}_{\alpha}, Q^{B}_{\beta}\} = 0$ of supersymmetric transformations of the component fields in $\mathcal{W}^{AB}$ can be still closed without any constraints on the component fields.

2.2 The structure of the stress-tensor supermultiplet in $\mathcal{N}=4$ coordinate superspace

Let us now briefly discuss the structure of the stress-tensor supermultiplet and the half-BPS multiplets in $\mathcal{N}=4$ superspace. The half-BPS multiplets by definition are annihilated by half of supersymmetry generators of the theory. The simplest example of the

half-BPS operators in the $\mathcal{N}=4$ SYM theory is the lowest component operators in the supermultiplet which consists of n identical scalars

$$\mathcal{O}^{(n)}_{AB} = Tr\,(\phi^n_{AB})\,. \tag{2.17}$$

In the case of $n=2$ the operator $\mathcal{O}^{(2)}_{AB}$ belongs also to the stress-tensor supermultiplet T^{AB}. This supermultiplet contains the stress-tensor and all other conserved currents in the $\mathcal{N}=4$ SYM theory. The stress-tensor supermultiplet is realized as a superfield T^{AB} in $\mathcal{N}=4$ coordinate superspace[5]

$$T^{AB} = \mathrm{Tr}\left(W^{AB}W^{AB}\right). \tag{2.18}$$

The following notation $T^{(0)}_{AB} = Tr(\phi^2_{AB})$ is used for the lowest component of T_{AB}. The superfield T^{AB} is not chiral in contrast to $\langle\Omega_n|$ superstate. However, one can restrict oneself to the chiral sector $\mathcal{T}^{AB}$ of T^{AB} which is realized for the particular projection $\mathcal{T}^{ab}$ of $\mathcal{T}^{AB}$ as

$$\mathcal{T}^{ab}(x,\theta^c) = Tr(\mathcal{W}^{ab}\mathcal{W}^{ab})(x,\theta^c),\ \mathcal{T}^{ab}(x,0) = Tr((\phi^{ab})^2). \tag{2.19}$$

All the component fields in $\mathcal{T}^{ab}$, as was explained earlier, are off-shell and $\mathcal{T}^{ab}$ is a pure chiral object.

Using the supercharges as translation generators in superspace, one can write $\mathcal{T}^{ab}$ as

$$\mathcal{T}^{ab}(x,\theta^c) = \exp(\theta^\alpha_c Q^c_\alpha)\mathcal{T}^{ab}(x,0). \tag{2.20}$$

One would like to note that the self-duality constraint implies $T^{(0)ab} = T^{(0)}_{\dot{a}\dot{b}}$.

Let us write explicitly the condition that operator $T^{(0)}_{\dot{a}\dot{b}}$ as the member of the half-BPS multiplet is annihilated by half of the SUSY generators. For $T^{(0)}_{\dot{a}\dot{b}}$ being a projection of $T^{(0)}_{AB}$ one splits supersymmetry generators as

$$\begin{aligned} &Q^A_\alpha \to (Q^a_\alpha|Q^{\dot{a}}_\alpha);\\ &\bar{Q}_{A\dot{\alpha}} \to (\bar{Q}_{a\dot{\alpha}}|\bar{Q}_{\dot{a}\dot{\alpha}}), \end{aligned} \tag{2.21}$$

and skipping the Lorentz indices one has

$$[Q^{\dot{c}}, T^{(0)}_{\dot{a}\dot{b}}] = 0,\ [\bar{Q}_c, T^{(0)}_{\dot{a}\dot{b}}] = 0. \tag{2.22}$$

2.3 The MHV superamplitudes at the tree level

Let us briefly discuss the amplitudes in on-shell momentum superspace. The symmetry arguments and the component answer for the tree-level gluon MHV amplitudes completely fix the MHV part of the superamplitude (2.5) at the tree level. One can use it for the

[5]more accurately, one has to write $T^{ABCD} = \mathrm{Tr}\left(W^{AB}W^{CD} - \frac{1}{24}\epsilon^{ABCD}\epsilon_{IKLM}W^{IK}W^{LM}\right)$.

loop computations by means of the unitarity based method and/or for the recursion relations to construct the non-MHV tree amplitudes. The explicit form of the MHV tree amplitude is important since the unitarity cut constructibility technique uses it as an input for reconstruction of the loop amplitudes.

The superamplitude must be invariant under the transformations generated by the full supersymmetry algebra in general, and in particular under the (super)translations (2.3) $p_{\alpha\dot\alpha},\ q^A_\alpha,\ \bar q_{A\dot\alpha}$ which imply

$$p_{\alpha\dot\alpha}\mathcal{A}_n = q^A_\alpha\mathcal{A}_n = \bar q_{A\dot\alpha}\mathcal{A}_n = 0, \tag{2.23}$$

where

$$p_{\alpha\dot\alpha} = \sum_{i=1}^n \lambda^i_\alpha\tilde\lambda^i_{\dot\alpha},\ q^A_\alpha = \sum_{i=1}^n \lambda^i_\alpha\eta^A_i,\ \bar q_{A\dot\alpha} = \sum_{i=1}^n \tilde\lambda^i_{\dot\alpha}\frac{\partial}{\partial\eta^A_i}.$$

From the above requirements one finds out that [23]

$$\mathcal{A}_n(\lambda,\tilde\lambda,\eta) = \delta^4(p_{\alpha\dot\alpha})\delta^8(q^A_\alpha)\mathcal{P}_n(\lambda,\tilde\lambda,\eta). \tag{2.24}$$

Here q^A_α can be understood as a superpartner of momentum $p_{\alpha\dot\alpha}$. The fact that $\mathcal{A}_n \sim \delta^4(\ldots)\delta^8(\ldots)$ is a reflection of supermomentum conservation, i.e., translation invariance in momentum superspace. The Grassmannian delta-function $\delta^8(q^A_\alpha)$ is defined by

$$\delta^8(q^A_\alpha) = \sum_{i,j=1}^n \prod_{A,B=1}^4 \langle ij\rangle\eta^A_i\eta^B_j. \tag{2.25}$$

The amplitude $\mathcal{A}_n$ is a polynomial in η of the order of $4n-8$ [23], $\delta^8(q^A_\alpha)$ is a polynomial in η of the order 8. Hence, if one expands $\mathcal{P}_n$ in powers of η, one obtains

$$\mathcal{P}_n = \mathcal{P}^{(0)}_n + \mathcal{P}^{(4)}_n + \ldots + \mathcal{P}^{(4n-16)}_n, \tag{2.26}$$

where $\mathcal{P}^{(4m)}_n$ is a homogenous $SU(4)_R$ invariant polynomial of the order of $4m$ (it can be deduced from the requirement $\bar q_{B\dot\alpha}\mathcal{A}_n = 0$). If one assigns the helicity $\lambda = +1$ to $|\Omega_i\rangle$ and $\lambda = +1/2$ to η then the amplitude $\mathcal{A}_n$ has an overall helicity $\lambda_\Sigma = n$, $\delta^8(q^A_\alpha)$ has helicity $\lambda_\Sigma = 4$ so that $\mathcal{P}^{(0)}_n$ has helicity $\lambda_\Sigma = n-4$. The MHV superamplitude is

$$\mathcal{A}^{MHV}_n(\lambda,\tilde\lambda,\eta) = \delta^4(p_{\alpha\dot\alpha})\delta^8(q^A_\alpha)\mathcal{P}^{(0)}_n(\lambda,\tilde\lambda). \tag{2.27}$$

The coefficients in the expansion of $\mathcal{A}^{MHV}_n$ with respect to the Grassmannian variables η^A_i are the component amplitudes. For example,

$$\mathcal{A}^{MHV}_n(\lambda,\tilde\lambda,\eta) = (\eta^1\eta^2\eta^3\eta^4)_1(\eta^1\eta^2\eta^3\eta^4)_2\langle g^-_1 g^-_2 g^+_3\ldots g^+_n\rangle + \ldots,$$

where $\langle g^-_1 g^-_2 g^+_3\ldots g^+_n\rangle$ is the MHV gluon component (Parke-Taylor) amplitude with all but two gluons with positive helicities. To extract the component amplitude from $\mathcal{A}^{MHV}_n$,

it is convenient to define the projecting operators

$$
\begin{aligned}
&|g^+\rangle: \ D_i^+ = 1, \\
&|\Gamma^A\rangle: \ D_i^A = \partial_i^A, \\
&|\phi^{AB}\rangle: \ D_i^{AB} = \frac{1}{2}\partial_i^A\partial_i^B, \\
&|\bar{\Gamma}^A\rangle: \ \bar{D}_i^D = \frac{1}{3!}\varepsilon^{ABCD}\partial_{iA}\partial_{iB}\partial_{iC}, \\
&|g^-\rangle: \ D_i^- = \frac{1}{4!}\varepsilon^{ABCD}\partial_{iA}\partial_{iB}\partial_{iC}\partial_{iD},
\end{aligned} \tag{2.28}
$$

which are the projectors on $|g^+\rangle$,$|\Gamma^A\rangle$,$|\phi^{AB}\rangle$,$|\bar{\Gamma}^A\rangle$,$|g^-\rangle$ states, respectively. The label i specifies the momentum of the state. All the previous discussions are valid for both the tree and the loop amplitudes.

One can determine the precise form of $\mathcal{P}_n^{(0)}$ for the tree amplitudes using the knowledge of the one particular component amplitude (for example, the Parke-Taylor one) [23]. Acting by the corresponding projection operators (2.28) on the MHV part of the superamplitude

$$
\begin{aligned}
&\langle g_1^- g_2^- g_3^+ \dots g_n^+ \rangle^{tree} = D_1^- D_2^- D_3^+ \dots D_n^+ \mathcal{A}_n^{tree,\ MHV}|_{\eta=0} = \\
&(\partial^1\partial^2\partial^3\partial^4)_1(\partial^1\partial^2\partial^3\partial^4)_2\delta^4(p_{\alpha\dot{\alpha}})\delta^8(q_\alpha^A)\mathcal{P}_n^{(0)}|_{\eta=0} = \delta^4(p_{\alpha\dot{\alpha}})\langle 12\rangle^4\mathcal{P}_n^{(0)},
\end{aligned} \tag{2.29}
$$

and comparing the result with the known component answer

$$
\langle g_1^- g_2^- g_3^+ \dots g_n^+ \rangle^{tree} = \delta^4(p_{\alpha\dot{\alpha}})\frac{\langle 12\rangle^4}{\langle 12\rangle \dots \langle n1\rangle}, \tag{2.30}
$$

one gets

$$
\mathcal{P}_n^{(0)} = \frac{1}{\langle 12\rangle \dots \langle n1\rangle}. \tag{2.31}
$$

So for the MHV part of the superamplitude

$$
\mathcal{A}_n^{tree,\ MHV}(\lambda, \tilde{\lambda}, \eta) = \delta^4(p_{\alpha\dot{\alpha}})\delta^8(q_\alpha^A)\frac{1}{\langle 12\rangle \dots \langle n1\rangle}. \tag{2.32}
$$

For the computations using the unitarity based techniques for the MHV sub-sector and two-particle (iterated) cuts one needs only the tree MHV superamplitudes $\mathcal{A}_n^{MHV,\ tree}$. However, the tree MHV amplitudes are also important for other calculations involving the NMHV amplitudes [22]. The fact that for the MHV sector/two-particle (iterated) cuts only the MHV amplitudes contribute can be seen from the naïve counting of η's

$$
\mathcal{A}_4^{1-loop,MHV} \sim \eta^4\eta^4,\ dSLIPS_2 \sim \partial^8/\partial\eta^8,\ \ \mathcal{A}_4^{tree,MHV} \sim \eta^4\eta^4.
$$

Here the super-Lorentz invariant phase space $dSLIPS_n$ is defined as

$$dSLIPS_n^{l_1,\ldots,l_n} = \delta^4\left(\sum_{i=1}^{n} l_i + \sum_{k=1}^{m} p_k\right)\prod_{i=1}^{n} d^4\eta_i\ \delta^+(l_i^2)\frac{d^D l_i}{(2\pi)^{D-1}}, \tag{2.33}$$

where $l_i, i = 1,\ldots,n$ are the momenta crossed by the cut with the associated Grassmannian variables η_i^A and $p_k, k = 1,\ldots,m$ are the external momenta.

For example, for the one-loop two-particle cut in the $s_{ij} = (p_i + p_j)^2$ channel one has

$$disc_{s_{ij}}[\mathcal{A}_4^{1-loop,MHV}] = \int dSLIPS_2^{l_1 l_2}\mathcal{A}_4^{tree,MHV}\mathcal{A}_4^{tree,MHV}. \tag{2.34}$$

The same is true for the iterated cuts with tree amplitudes at higher order of PT. The latter fact resembles the MHV vertex expansion method [5], where only the MHV amplitudes as effective vertices in the diagrams are used. In the on-shell momentum superspace for the two-particle cuts of the MHV part of the amplitude the following formula for summation over the states in the cut (Grasmannian integration in $dSLIPS$) can be applied

$$\int d^4\eta_{l_1}d^4\eta_{l_2}\delta^8\left(\lambda_\alpha^{l_1}\eta_{l_1}^A + \lambda_\alpha^{l_2}\eta_{l_2}^A + Q_\alpha^A\right)\delta^8\left(\lambda_\alpha^{l_1}\eta_{l_1}^A + \lambda_\alpha^{l_2}\eta_{l_2}^A - P_\alpha^A\right)$$
$$= \langle l_1 l_2\rangle^4\delta^8\left(P_\alpha^A + Q_\alpha^A\right). \tag{2.35}$$

Similar results are also obtained for the amplitudes in the theories with lower supersymmetry ($1 \le \mathcal{N} < 4$) [25]. For example, one can write an analog of (2.35) in theories with $1 \le \mathcal{N} < 4$ supersymmetry [25]

$$\int d^{\mathcal{N}}\eta_{l_1}d^{\mathcal{N}}\eta_{l_2}\delta^{2\mathcal{N}}\left(\lambda_\alpha^{l_1}\eta_{l_1}^A + \lambda_\alpha^{l_2}\eta_{l_2}^A + Q_\alpha^A\right)\delta^{2\mathcal{N}}\left(\lambda_\alpha^{l_1}\eta_{l_1}^A + \lambda_\alpha^{l_2}\eta_{l_2}^A - P_\alpha^A\right)$$
$$= \langle l_1 l_2\rangle^{\mathcal{N}}\delta^{2\mathcal{N}}\left(P_\alpha^A + Q_\alpha^A\right), \tag{2.36}$$

where A is the R-symmetry group index (for $SU(2)_R$ and $SU(4)_R$ in case of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SUSY, respectively). The definition of the Grassmannian delta-function $\delta^{2\mathcal{N}}(\ldots)$ for $\mathcal{N} = 4$ is given by (2.25), and for $\mathcal{N} = 2$ the definition is presented further in the text [25].

It is possible to write an on-shell momentum superspace generalization for form factors of chiral truncation for the stress-tensor supermultiplet T^{AB}. Moreover, using this formulation the supersums in unitarity based loop computations for super form factors in the MHV sector can be performed in a similar way as for the amplitudes.

2.4 Superstate – form factor

To construct "superstate – form factor" as a generalization of component form factors with the $T_{AB}^{(0)}$ operator, one begins with the observation that for the form factor of the

operator $T^{(0)}_{AB}$ with external state made of gluons with + polarization and 2 scalars $\overline{\phi_{AB}}$ the answer is known at the tree level [14]. For some particular choice of A, B (let us take $\dot{a}, \dot{b}$) one gets

$$\langle g_1^+ \dots \phi_{i\ ab} \dots \phi_{j\ ab} \dots g_n^+ | T^{(0)}_{\dot{a}\dot{b}}(q)|0\rangle = \delta^4(\sum_{l=1}^{n} \lambda_\alpha^l \tilde{\lambda}_{\dot{\alpha}}^l - q_{\alpha\dot{\alpha}}) \frac{\langle ij\rangle^2}{\langle 12\rangle \dots \langle n1\rangle}, \tag{2.37}$$

where q is arbitrary momentum carried by the $T^{(0)}_{\dot{a}\dot{b}}$ operator.

One considers a generalization of this result to the object of the form

$$F_n(\{\lambda, \tilde{\lambda}, \eta\}, q) = \langle \Omega_n | T^{(0)}_{\dot{a}\dot{b}}(q)|0\rangle, \tag{2.38}$$

where we keep the operator in component form and promote the external state to its superversion (2.4)

$$\langle \Omega_n | = \langle 0 | \prod_{i=1}^{n} \Omega_i. \tag{2.39}$$

Using the fact that the supercharges annihilate the vacuum and the commutation relations (2.22) one gets

$$\langle \Omega_n | Q_\alpha^{\dot{c}} T^{(0)}_{\dot{a}\dot{b}}|0\rangle = \langle \Omega_n | \bar{Q}_{c\dot{\alpha}} T^{(0)}_{\dot{a}\dot{b}}|0\rangle = 0. \tag{2.40}$$

In combination with momentum conservation this leads to the following properties of the superstate form factor F_n [6]:

$$q_\alpha^{\dot{a}} F_n = \bar{q}_{\dot{\alpha}a} F_n = p_{\alpha\dot{\alpha}} F_n = 0, \tag{2.41}$$

where

$$q_\alpha^{\dot{a}} = \sum_{i=1}^{n} \lambda_\alpha^i \eta_i^{\dot{a}},\ \ \bar{q}_{a\dot{\alpha}} = \sum_{i=1}^{n} \tilde{\lambda}_{\dot{\alpha}}^i \frac{\partial}{\partial \eta_i^a},\ \ p_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \lambda_\alpha^i \tilde{\lambda}_{\dot{\alpha}}^i - q_{\alpha\dot{\alpha}}. \tag{2.42}$$

Then one writes, in full analogy with superamplitude (2.24), the expression for the super form factor

$$F_n(\{\lambda, \tilde{\lambda}, \eta\}, q) = \delta^4(\sum_{i=1}^{n} \lambda_\alpha^i \tilde{\lambda}_{\dot{\alpha}}^i - q_{\alpha\dot{\alpha}}) \delta^4_{GR}\left(q_\alpha^{\dot{a}}\right) \left(\mathcal{X}_n^{(0)} + \mathcal{X}_n^{(4)} + \dots + \mathcal{X}_n^{(4n-8)}\right) \tag{2.43}$$

where $\mathcal{X}_n^{(4m)}$ are the homogenous $SU(4)_R$ invariant polynomials of the order of $4m$. The Grassmannian delta-function δ^4_{GR} is defined as (one uses the GR subscript to distinguish it from the ordinary bosonic delta-function)

$$\delta^4_{GR}\left(\sum_{i=1}^{n} \lambda_\alpha^i \eta_i^{\dot{a}}\right) = \sum_{i,j=1}^{n} \prod_{\dot{a},\dot{b}=1,2} \langle ij\rangle \eta_i^{\dot{a}} \eta_j^{\dot{b}}. \tag{2.44}$$

[6]We thank A. Zhiboedov for discussion of this point.

Here in contrast to the expression (2.26) for the amplitudes where the expansion was up to the polynomial $\mathcal{P}_n^{(4n-16)}$ one has only the expansion up to $\mathcal{X}_n^{(4n-8)}$ associated with the difference in the number of super charges which annihilate F_n.

If one, as in the case of superamplitude, assigns helicity $\lambda = +1$ to $|\Omega_i\rangle$ and $\lambda = +1/2$ to η, one sees that F_n has an overall helicity $\lambda_\Sigma = n$, δ^4_{GR} has $\lambda_\Sigma = 2$ so that $\mathcal{X}_n^{(0)}$ has $\lambda_\Sigma = n - 2$ which is understood as an analog of the MHV part of superamplitude (2.5)

$$F_n^{MHV}(\{\lambda, \tilde{\lambda}, \eta\}, q) = \delta^4(\sum_{i=1}^{n} \lambda_\alpha^i \tilde{\lambda}_{\dot{\alpha}}^i - q_{\alpha\dot{\alpha}})\delta^4_{GR}\left(q_\alpha^{\dot{a}}\right) \mathcal{X}_n^{(0)}(\lambda, \tilde{\lambda}). \tag{2.45}$$

The non-MHV contributions (N^{k}MHV) to $F_n(\{\lambda, \tilde{\lambda}, \eta\}, q)$ are not considered in this paper. To obtain the component answer from the latter expression, the projection operators (2.28) are considered in full analogy with the superamplitude. The overall order of projectors in this case is 4 instead 8 for the superamplitude. At the tree level, comparing this result with the component answer (2.37) one obtains

$$\mathcal{X}_n^{(0)} = \frac{1}{\langle 12\rangle \ldots \langle n1\rangle}. \tag{2.46}$$

Being manifestly not $SU(4)_R$ covariant δ^4_{GR} is not suitable for performing the supersummations since it depends only on half of η variables. In principle, one can formally write δ^4_{GR} as

$$\delta^4_{GR}(q_\alpha^{\dot{a}}) = \frac{(\partial^a\partial^b)'(\partial^a\partial^b)''}{\langle\lambda'\lambda''\rangle^2}\delta^8(q_\alpha^A + \lambda'_\alpha\eta'^A + \lambda''_\alpha\eta''^A)|_{\eta'=\eta''=0}. \tag{2.47}$$

Defining a projection operator

$$\Pi^{\dot{a}\dot{b}} = \frac{(\partial^a\partial^b)'(\partial^a\partial^b)''}{\langle\lambda'\lambda''\rangle^2}|_{\eta'=\eta''=0}, \tag{2.48}$$

one can then write the MHV part of the form factor at the tree level as

$$F_n^{MHV,tree}(\{\lambda, \tilde{\lambda}, \eta\}, q) = \delta^4(\sum_{i=1}^{n} \lambda_\alpha^i \tilde{\lambda}_{\dot{\alpha}}^i - q_{\alpha\dot{\alpha}})\frac{\Pi^{\dot{a}\dot{b}}\delta^8\left(q_\alpha^A\right)}{\langle 12\rangle \ldots \langle n1\rangle}, \tag{2.49}$$

where now

$$q_\alpha^A = \sum_{i=1}^{n} \lambda_\alpha^i \eta_i^A + \lambda'_\alpha\eta'^A + \lambda''_\alpha\eta''^A.$$

For the form factor written in such a form the two-particle supersums for the MHV sector are performed in the same fashion as for the amplitudes since the auxiliary variables η', η'' are always harmless and the projection operator $\Pi^{\dot{a}\dot{b}}$ stays outside the Grassmannian integral in $dSLIPS$.

Acting on (2.49) with the projection operators (2.28) to external state one obtains the component form factors. For example, the form factor of $T^{(0)}_{\dot{a}\dot{b}}$ with external state consisting of the gluons with + polarization, 1 scalar and 2 fermions is given by

$$\langle g_1^+ \ldots \phi_{i\ ab} \ldots \psi_{j\ a} \ldots \psi_{k\ b} \ldots g_n^+ | T^{(0)}_{\dot{a}\dot{b}}(q)|0\rangle = \delta^4(\sum_{l=1}^{n} \lambda^l_\alpha \tilde{\lambda}^l_{\dot{\alpha}} - q_{\alpha\dot{\alpha}}) \frac{\langle ij\rangle\langle ik\rangle}{\langle 12\rangle \ldots \langle n1\rangle}.$$

For a small number of particles in external state ($n = 3$, for example) one easily verifies this result by direct diagram computation. Formally, to obtain this answer for different values of $SU(4)_R$ indices of external state particles one has to consider different projections; however, this expression is independent of the particular values of the $SU(4)_R$ index.

In principle, expression (2.49) can be used in the unitarity based computations as a building block together with $\mathcal{A}_n^{MHV,tree}$. However, it is desirable to obtain the formulation of super form factors where the operator is also promoted to its supersymmetric version. The next subsection is devoted to such generalization.

2.5 Superstate – super form factor

It is natural to consider the operator $T^{(0)}_{\dot{a}\dot{b}}$ as the first term in series expansion in θ's of the stress-tensor supermultiplet T^{AB} for a particular choice of A and B. However, as was mentioned before, T^{AB} is not a chiral superfield in contrast to the superstate $\langle\Omega_n|$ and it is not convenient to consider the objects of different chirality types. Also, the component fields in T^{AB} are constrained, while for insertion of an operator a pure off-shell object is preferred. So one restricts oneself to the superfield $\mathcal{T}^{ab}$ instead of T^{AB}; $T^{(0)}_{\dot{a}\dot{b}}$ is also the first term of the expansion in θ's of $\mathcal{T}^{ab}$ which contains in particular the chiral part of projection (2.11), (2.12) of the supermultiplet T^{AB} and is closed under the chiral $\mathcal{N} = 4$ supersymmetric transformations. Here $\mathcal{T}^{ab}$ is used as the operator insertion. The generalization of (2.38) would be the replacement of $T^{(0)}_{\dot{a}\dot{b}}$ in (2.38) by $\mathcal{T}^{ab}$. This is because all the fields in $\mathcal{T}^{ab}$ are still unconstrained (off-shell).

Thus, we consider a more general object different from (2.38) of the following form:

$$\mathcal{F}_n(\{\lambda, \tilde{\lambda}, \eta\}, q, \theta^a) = \langle\Omega_n|\mathcal{T}^{ab}(q, \theta^a)|0\rangle.$$

Remind that $\mathcal{F}_n$ depends only on $\theta^a, a = 1, 2$ and not on $\theta^{\dot{a}}, \dot{a} = 1, 2$. Using (2.20) we can rewrite it explicitly as

$$\mathcal{F}_n(\{\lambda, \tilde{\lambda}, \eta\}, q, \theta^a) = \exp(\theta^\alpha_a q^a_\alpha)\langle\Omega_n|T^{(0)}_{\dot{a}\dot{b}}(q)|0\rangle = \exp(\theta^\alpha_a q^a_\alpha) F_n(\{\lambda, \tilde{\lambda}, \eta\}, q), \qquad (2.50)$$

where $q^a_\alpha = \sum_{i=1}^n \lambda^i_\alpha \eta^a$. Hence, following the discussion in the previous section one gets

$$\mathcal{F}_n(\{\lambda, \tilde{\lambda}, \eta\}, q, \theta^a) = e^{\theta^\alpha_a q^a_\alpha} \delta^4(\sum_{i=1}^{n} \lambda^i_\alpha \tilde{\lambda}^i_{\dot{\alpha}} - q_{\alpha\dot{\alpha}}) \delta^4_{GR}\left(q^{\dot{a}}_\alpha\right) \left(\mathcal{X}_n^{(0)} + \mathcal{X}_n^{(4)} + \ldots + \mathcal{X}_n^{(4n-8)}\right). \qquad (2.51)$$

We see that the generalization of the previous results to the "superstate– super form factor" is straightforward. Let us make a comment about the helicity structure of $\mathcal{F}_n$. If we assign helicity $\lambda = -1/2$ to θ^a_α then the exponential factor carries the total helicity $\lambda_\Sigma = 0$. Using the arguments from the previous subsection we conclude that $\mathcal{X}^{(0)}_n$ still has the total helicity $\lambda_\Sigma = n-2$ and it is understood as an analog of the MHV part for the superamplitude. At the tree level, comparing the obtained expression with the component answer (2.50) and identifying the first term in θ expansion with (2.50) one obtains $\mathcal{X}^{(0)}_n = 1/\langle 12\rangle \ldots \langle n1\rangle$ so that for the MHV sector at the tree level one gets

$$\mathcal{F}_n^{MHV,tree}(\{\lambda,\tilde{\lambda},\eta\},q,\theta^a) = \delta^4(\sum_{i=1}^n \lambda^i_\alpha\tilde{\lambda}^i_{\dot{\alpha}} - q_{\alpha\dot{\alpha}})\exp(\theta^\alpha_a q^a_\alpha)\frac{\delta^4_{GR}\left(q^{\dot{a}}_\alpha\right)}{\langle 12\rangle\ldots\langle n1\rangle}. \qquad (2.52)$$

Note that similar structures, namely, the exponent $\exp(\theta^\alpha_a q^a_\alpha)$ in the context of form factors in $\mathcal{N}=4$ SYM first appeared in [12] where the non-gauge invariant form factors ("off shell currents") of the form $\langle\Omega_n|W^{AB}|0\rangle$ were studied.

Expanding $\exp(\theta^\alpha_a q^a_\alpha)$ in powers of θ one gets for the m's ($m \leq 4$) coefficient of expansion of the form factors in the form $\langle\Omega_n|T^{ab}_{(m)}|0\rangle$, where $T^{ab}_{(m)}$ are the operators which belong to the chiral part of the supermultiplet T^{AB}. For example, the operators that are the Lorentz scalars have the following form:

$$T^{ab}_{(0)} = Tr(\phi^2_{\dot{a}\dot{b}}),\ T^{ab}_{(2)} = Tr(\psi^a_\alpha\psi^{a\alpha}),\ T^{ab}_{(4)} = Tr(F^{\alpha\beta}F_{\alpha\beta}) + O(g). \qquad (2.53)$$

One can also assign a projection operator of the form $\partial^m_\theta|_{\theta=0}$ to each of the form factors $\langle\Omega_n|T^{ab}_{(m)}|0\rangle$. If one wants to obtain operators with different values of indices from the chiral part of T^{AB}, one has to consider different projections.

Let us now discuss the properties of $\mathcal{F}_n^{tree,MHV}$ under supersymmetric transformations in more detail. The part of supersymmetry generators (2.7) which acts on $\mathcal{T}^{ab}$ can be written as

$$\begin{aligned}
\text{4 translations } P_{\alpha\dot{\alpha}} &= q_{\alpha\dot{\alpha}},\\
\text{8 supercharges } Q^A_\alpha &= \frac{\partial}{\partial\theta^\alpha_A},\\
\text{8 conjugated supercharges } \bar{Q}_{A\dot{\alpha}} &= \theta^\alpha_A q_{\alpha\dot{\alpha}},
\end{aligned} \qquad (2.54)$$

while the supersymmetry generators acting on $\mathcal{F}_n^{tree,MHV}$ are

$$\begin{aligned}
\text{4 translations } P_{\alpha\dot{\alpha}} &= -\sum_{i=1}^n \lambda^i_\alpha\tilde{\lambda}^i_{\dot{\alpha}} + q_{\alpha\dot{\alpha}},\\
\text{4 supercharges } Q^a_\alpha &= -\sum_{i=1}^n \lambda^i_\alpha\eta^a_i + \frac{\partial}{\partial\theta^\alpha_a},\\
\text{4 supercharges } Q^{\dot{a}}_\alpha &= -\sum_{i=1}^n \lambda^i_\alpha\eta^{\dot{a}}_i + \frac{\partial}{\partial\theta^\alpha_{\dot{a}}},
\end{aligned}$$

$$
\begin{aligned}
\text{4 conjugated supercharges } \bar{Q}_{a\dot\alpha} &= -\sum_{i=1}^{n} \tilde\lambda^i_{\dot\alpha} \frac{\partial}{\partial \eta_i^a} + \theta_a^\alpha q_{\alpha\dot\alpha}, \\
\text{4 conjugated supercharges } \bar{Q}_{\dot a\dot\alpha} &= -\sum_{i=1}^{n} \tilde\lambda^i_{\dot\alpha} \frac{\partial}{\partial \eta_i^{\dot a}} + \theta_{\dot a}^\alpha q_{\alpha\dot\alpha}.
\end{aligned} \tag{2.55}
$$

The action of the supersymmetric generators Q^a_α and $Q^{\dot a}_\alpha$ on $\mathcal{F}_n^{tree,MHV}$ is given by

$$
\begin{aligned}
Q^a_\alpha \mathcal{F}_n^{tree,MHV} &= \delta^4(\sum_{i=1}^{n} \lambda^i_\alpha \tilde\lambda^i_{\dot\alpha} - q_{\alpha\dot\alpha}) \left(Q^a_\alpha e^{\theta_c^\beta q_\beta^c} \right) \frac{\delta^4_{GR}(\sum_{i=1}^n \lambda^i_\alpha \eta_i^{\dot a})}{\langle 12\rangle \dots \langle n1\rangle} = \\
&= \delta^4(\sum_{i=1}^{n} \lambda^i_\alpha \tilde\lambda^i_{\dot\alpha} - q_{\alpha\dot\alpha}) \left(-\sum_{i=1}^{n} \lambda^i_\alpha \eta_i^a + \sum_{i=1}^{n} \lambda^i_\alpha \eta_i^a \right) e^{\theta_c^\beta q_\beta^c} \frac{\delta^4_{GR}(\sum_{i=1}^n \lambda^i_\alpha \eta_i^{\dot a})}{\langle 12\rangle \dots \langle n1\rangle} = 0,
\end{aligned} \tag{2.56}
$$

$$
Q^{\dot a}_\alpha \mathcal{F}_n^{tree,MHV} = \delta^4(\sum_{i=1}^{n} \lambda^i_\alpha \tilde\lambda^i_{\dot\alpha} - q_{\alpha\dot\alpha}) \left(-\sum_{i=1}^{n} \lambda^i_\alpha \eta_i^{\dot a} \right) \delta^4_{GR}(\sum_{i=1}^{n} \lambda^i_\alpha \eta_i^{\dot a}) \frac{e^{\theta_c^\beta q_\beta^c}}{\langle 12\rangle \dots \langle n1\rangle} = 0. \tag{2.57}
$$

From (2.56) and (2.57) one observes that $\mathcal{F}_n^{tree,MHV}$ is invariant under the full chiral part of supersymmetry generators Q^A_α

$$
Q^A_\alpha \mathcal{F}_n^{tree,MHV} = 0.
$$

This might be little surprising since one expects that the half-BPS objects should be annihilated only by half of chiral and antichiral supercharges. Heuristically, this can be explained as reflection of the fact that $\mathcal{F}_n$ is not only a half-BPS object but also contains an operator from chiral projection of T^{AB} which should be closed under the chiral part of supersymmetry.

The action of antichiral supersymmetric generators is

$$
\begin{aligned}
\bar{Q}_{a\dot\alpha} \mathcal{F}_n^{tree,MHV} &= \delta^4(\sum_{i=1}^{n} \lambda^i_\alpha \tilde\lambda^i_{\dot\alpha} - q_{\alpha\dot\alpha}) \left(\bar{Q}_{a\dot\alpha} e^{\theta_c^\beta q_\beta^c} \right) \frac{\delta^4_{GR}(\sum_{i=1}^n \lambda^i_\alpha \eta_i^{\dot a})}{\langle 12\rangle \dots \langle n1\rangle} = \\
& \delta^4(\sum_{i=1}^{n} \lambda^i_\alpha \tilde\lambda^i_{\dot\alpha} - q_{\alpha\dot\alpha}) \ \left(-\sum_{i=1}^{n} \tilde\lambda^i_{\dot\alpha} \theta_a^\beta \lambda^i_\beta + \theta_a^\beta q_{\beta\dot\alpha} \right) e^{\theta_c^\beta q_\beta^c} \frac{\delta^4_{GR}(\sum_{i=1}^n \lambda^i_\alpha \eta_i^{\dot a})}{\langle 12\rangle \dots \langle n1\rangle} = \\
& \delta^4(\sum_{i=1}^{n} \lambda^i_\alpha \tilde\lambda^i_{\dot\alpha} - q_{\alpha\dot\alpha}) \sum_{i=1}^{n} \theta_a^\beta \left(-\tilde\lambda^i_{\dot\alpha} \lambda^i_\beta + \tilde\lambda^i_{\dot\alpha} \lambda^i_\beta \right) e^{\theta_c^\beta q_\beta^c} \frac{\delta^4_{GR}(\sum_{i=1}^n \lambda^i_\alpha \eta_i^{\dot a})}{\langle 12\rangle \dots \langle n1\rangle} = 0,
\end{aligned} \tag{2.58}
$$

while for $\bar{Q}_{\dot a\dot\alpha}$

$$
\begin{aligned}
\bar{Q}_{\dot a\dot\alpha} \mathcal{F}_n^{tree,MHV} &= \delta^4(\sum_{i=1}^{n} \lambda^i_\alpha \tilde\lambda^i_{\dot\alpha} - q_{\alpha\dot\alpha}) \left(\bar{Q}_{\dot a\dot\alpha} \delta^4_{GR}(\sum_{i=1}^{n} \lambda^i_\alpha \eta_i^{\dot a}) \right) \frac{e^{\theta_a^\beta q_\beta^a}}{\langle 12\rangle \dots \langle n1\rangle} = \\
\delta^4(\sum_{i=1}^{n} \lambda^i_\alpha \tilde\lambda^i_{\dot\alpha} - q_{\alpha\dot\alpha}) & \left(-q_{\alpha\dot\alpha} + \delta^4_{GR}(\sum_{i=1}^{n} \lambda^i_\alpha \eta_i^{\dot a}) \theta_{\dot a}^\beta q_{\beta\dot\alpha} \right) \frac{e^{\theta_a^\beta q_\beta^a}}{\langle 12\rangle \dots \langle n1\rangle} \neq 0.
\end{aligned} \tag{2.59}
$$

So $\mathcal{F}_n^{tree,MHV}$ is invariant under half of the antichiral supersymmetry generators $\bar{Q}_{a\dot{\alpha}}$, which reflects the fact that $\mathcal{F}_n^{tree,MHV}$ is a half-BPS object.

$\mathcal{F}_n^{MHV,tree}$ being a pure chiral object can be rewritten in the form which resembles the superamplitude given by (2.24). For this one should define the transformation in the same spirit as [26], but in this case the logic is inverted, and one replaces a "physical" coordinate θ with the set of auxiliary variables (moduli) $\{\lambda, \eta\}$

$$\hat{T}[\ldots] = \int d^4\theta_\alpha^a \exp(\theta_a^\alpha \sum_{i=1}^{n} \lambda_\alpha^i \eta_i^a)[\ldots]. \tag{2.60}$$

For $n = 2$ (for $n = 1$ the Grassmannian delta function vanishes) one has

$$\hat{T}[1] = \delta^4_{GR}(\sum_{i=1}^{2} \lambda_\alpha^i \eta_i^a), \tag{2.61}$$

and applying the transformation $\hat{T}$ to $\mathcal{F}_n^{tree,MHV}$

$$\begin{aligned} &Z_n^{tree,MHV}(\{\lambda, \tilde{\lambda}, \eta\}, q, \{\lambda', \lambda'', \eta^{'a}, \eta^{''a}\}) = \hat{T}[\mathcal{F}_n^{tree,MHV}] = \\ &\delta^4(\sum_{i=1}^{n} \lambda_\alpha^i \tilde{\lambda}_{\dot{\alpha}}^i - q_{\alpha\dot{\alpha}}) \frac{\delta^4_{GR}(q_\alpha^a + \lambda_\alpha^{'} \eta^{'a} + \lambda_\alpha^{''} \eta^{''a}) \delta^4_{GR}\left(q_\alpha^{\dot{a}}\right)}{\langle 12 \rangle \ldots \langle n1 \rangle}, \end{aligned} \tag{2.62}$$

where $\{\lambda', \lambda'', \eta^{'a}, \eta^{''a}\}$ are the auxiliary variables which replace the θ coordinate. This expression is still not exactly $SU(4)_R$ covariant. However, one can still perform the supersummation using (2.36). One can decompose the Grassmannian delta-function δ^8 in the tree MHV amplitude into the product of two δ^4_{GR} and the integration measure $d^4\eta^A \to d^2\eta^a d^2\eta^{\dot{a}}$ and then use the relation which consists of two copies of (2.36), where one has to put $\mathcal{N} = 2$ (which corresponds to braking of $SU(4)_R$ down to two $SU(2)$s in our case)

$$\begin{aligned} \int d^2\eta_{l_1}^a d^2\eta_{l_2}^a d^2\eta_{l_1}^{\dot{a}} d^2\eta_{l_2}^{\dot{a}} \delta^4_{GR}\left(\lambda_\alpha^{l_1}\eta_{l_1}^a + \lambda_\alpha^{l_2}\eta_{l_2}^a + Q_\alpha^a\right) \delta^4_{GR}\left(\lambda_\alpha^{l_1}\eta_{l_1}^{\dot{a}} + \lambda_\alpha^{l_2}\eta_{l_2}^{\dot{a}} + \tilde{Q}_\alpha^{\dot{a}}\right) \\ \times \delta^4_{GR}\left(\lambda_\alpha^{l_1}\eta_{l_1}^a + \lambda_\alpha^{l_2}\eta_{l_2}^a - P_\alpha^a\right) \delta^4_{GR}\left(\lambda_\alpha^{l_1}\eta_{l_1}^{\dot{a}} + \lambda_\alpha^{l_2}\eta_{l_2}^{\dot{a}} - P_\alpha^{\dot{a}}\right) \\ = \langle l_1 l_2 \rangle^4 \delta^4_{GR}\left(P_\alpha^a + Q_\alpha^a\right) \delta^4_{GR}\left(P_\alpha^{\dot{a}} + \tilde{Q}_\alpha^{\dot{a}}\right). \end{aligned} \tag{2.63}$$

There is also a possibility to obtain completely $SU(4)_R$ covariant expression for the MHV super form factors. All the solutions of Grassmannian analyticity constraints (2.13) can be combined together by introducing the $\mathcal{N} = 4$ harmonic superspace. In such a setup $SU(4)_R$ covariance is manifest in all expressions. We will give such a formulation of super form factor in appendix A. However, in unitarity based computations in the MHV sector for the form factors with operators from the chiral truncation of T^{AB} such a formulation does not give any significant computational simplifications. So we will use

$Z_n^{tree,MHV}$ further on instead of harmonic superspace one. In principle, the situation with the computation of more complicated objects (such as the NMHV super form factors, the operators from different supermultiplets, etc) can be different and harmonic superspace formulation might be useful.

The projection operators can be written as projectors in terms of $\eta_A^{'}, \eta_A^{''}$. For example, for operators (2.53) one gets the following projectors in terms of $\eta_A^{'}, \eta_A^{''}$'s, respectively,

$$\begin{aligned}
\hat{T}:\ 1|_{\theta=0} &\rightarrow \frac{(\partial^A\partial^B)^{'}(\partial^A\partial^B)^{''}}{\langle\lambda'\lambda''\rangle^2}|_{\eta'=\eta''=0},\\
\hat{T}:\ \partial_\theta^2|_{\theta=0} &\rightarrow \frac{(\partial^A)^{'}(\partial^A)^{''}}{\langle\lambda'\lambda''\rangle}|_{\eta'=\eta''=0},\\
\hat{T}:\ \partial_\theta^4|_{\theta=0} &\rightarrow 1|_{\eta'=\eta''=0}.
\end{aligned} \tag{2.64}$$

The algorithm for obtaining a particular component answer from $Z_n^{tree,MHV}$ can be formulated as follows. One has to apply a particular projector constructed from $\eta_A^{'}, \eta_A^{''}$ to fix the corresponding operator (like the one in (2.53)), and then apply a sufficient number of projectors (2.28) on the components in the $\langle\Omega_n|$ superstate. So the projector operator $\Pi^{\dot{a}\dot{b}}$ in (2.49) is understood as the projection of $Z_n^{tree,MHV}$ into the $\langle\Omega_n|T^{(0)}_{\dot{a}\dot{b}}|0\rangle$ form factor. As another example, one can obtain an expression for the form factor of $T^{ab}_{(2)} = Tr(\psi^a_\alpha\psi^{b\alpha})$ with external state consisting of gluons with + polarization, and 2 fermions

$$\langle g_1^+ \ldots \bar{\psi}_{i\ a} \ldots \bar{\psi}_{j\ b} \ldots g_n^+|T^{ab}_{(2)}(q)|0\rangle = \delta^4(\sum_{k=1}^n \lambda^k_\alpha\tilde{\lambda}^k_{\dot{\alpha}} - q_{\alpha\dot{\alpha}})\frac{\langle ij\rangle^3}{\langle 12\rangle\ldots\langle n1\rangle}.$$

For a small number of particles in external state ($n = 3$ for example) one can easily verify this result by direct diagram computation. Note that to obtain the expected total helicity $\lambda_\Sigma = n - 2$ for this form factor one also has to consider the total helicity of the operator.

2.6 Soft limit. From form factors to amplitudes

The form of super form factor $Z_n^{tree,MHV}$ (2.62) resembles the superamplitude (2.5), and moreover coincides with it in the soft limit (where supermomentum carried by an operator goes to 0)[7], i.e.

$$Z_n^{tree,MHV}(\{\lambda,\tilde{\lambda},\eta\},0,\{0\}) = \mathcal{A}_n^{tree,MHV}(\lambda,\tilde{\lambda},\eta). \tag{2.65}$$

The explanation of this fact is the following. It is known that derivative of the amplitude $A_n = \langle p_1^{\lambda_1}\ldots p_n^{\lambda_n}|0\rangle$ (or the correlation function of some local composite operator of the theory) with respect to the coupling constant g after the appropriate rescaling of the fields in the Lagrangian $\mathcal{L}$ is equivalent to insertion of the Lagrangian operator with zero

[7] We acknowledge the discussion with A. Zhiboedov who pointed out the latter fact to us.

momentum in the momentum representation (similar to duality between the correlation functions and the amplitudes in [17])

$$g\frac{\partial A_n}{\partial g} = \langle p_1^{\lambda_1} \dots p_n^{\lambda_n} | \mathcal{L}(q=0) | 0 \rangle. \tag{2.66}$$

On the other hand, the operator $T^{ab}_{(4)}$ of the $\mathcal{T}^{ab}$ which is a chiral part of T^{AB} supermultiplet is equal to the $\mathcal{N}=4$ Lagrangian written in the chiral form (see [24] for details)

$$T^{ab}_{(4)} = \mathcal{L}^{\mathcal{N}=4}_{chiral} = \int d^4\theta^c_\alpha \mathcal{T}^{ab}(q, \theta^c). \tag{2.67}$$

Using the fact that the Grassmannian integration is equivalent to differentiation and the properties (2.64) of $\hat{T}$ transformation one can write

$$\mathcal{L}^{\mathcal{N}=4}_{chiral}(q) = \hat{T}[\mathcal{T}^{ab}](q, 0), \tag{2.68}$$

which gives

$$Z_n^{MHV}(\{\lambda, \tilde{\lambda}, \eta\}, 0, \{0\}) = \hat{T}[\mathcal{F}_n^{MHV}](\{\lambda, \tilde{\lambda}, \eta\}, 0, \{0\}) = g\frac{\partial \mathcal{A}_n^{MHV}(\lambda, \tilde{\lambda}, \eta)}{\partial g}. \tag{2.69}$$

This leads to the following conjecture: *the MHV superstate super form factor with zero operator supermomentum is given by the logarithmic derivative of the MHV superamplitude with respect to the coupling constant.*

We will verify this relation at one loop in the MHV sector by direct computation further in the text. Note also that in the soft limit the action of $\bar{Q}_{\dot{\alpha}\dot{a}}$ on (2.59) gives 0, as it should be for the superamplitude. The conjectured relation, however, may be inconsistent for the N^{k}MHV sector due to the absence of a smooth limit for the N^{k}MHV form factors. Indeed it is known that NMHV amplitudes contain multiparticle poles. The NMHV form factors likely share the same property. For example, using the BCFW recursion relations one can obtain the four-point NMHV form factor [27]

$$\langle \phi_1^{AB} g_2^+ g_3^- \phi_4^{AB} | T^{(0)}_{AB} | 0 \rangle = \frac{\langle 13 \rangle [24]}{s_{23} \langle 12 \rangle [34]} + \frac{\langle 34 \rangle [24][24]}{s_{23} s_{234} [34]} + \frac{\langle 13 \rangle [12] \langle 13 \rangle}{s_{23} s_{123} \langle 12 \rangle},$$

where $s_{ij} = (p_i + p_j)^2, s_{ijk} = (p_i + p_j + p_k)^2$. One observes that in the limit $q \to 0$ the last two terms in the latter expression become singular due to the presence of s_{123} and s_{234} in the denominators.

3 Reduction procedure for dual pseudoconformal scalar integrals

The $D = 4$ unitarity based method proved to be an effective tool for the loop computations [7, 28, 29, 30]. Despite the fact that any ansatz obtained by the $D = 4$ unitary-based

method should be confirmed by some direct D-dimensional computation or additional information, the $D = 4$ methods can offer the crucial guidance for constructing a full D-dimensional answer [22]. In the $\mathcal{N} = 4$ SYM it is known that the $D = 4$ unitary-based methods gives a correct answer up to $O(\epsilon)$ at the one-loop order for the arbitrary number of external legs. It was also observed that it captures the correct answer for the four-point amplitude up to four loops [28] and for the five-point amplitude up to two loops [30] for the even part (i.e. part which does not contain γ^5) of the amplitude.

In a general (non)supersymmetric theory there is an unknown basis of scalar integrals beyond one loop. However, for the $\mathcal{N} = 4$ SYM theory the situation is different. It was observed [19, 20] that the amplitudes in the $\mathcal{N} = 4$ SYM possess a new type of symmetry - the dual conformal symmetry. The reflection of this symmetry on the level of scalar integrals is that all scalar integrals entering into the final answer should be pseudoconformal integrals in momentum space[8]. This, in fact, defines the basis of scalar integrals for the $\mathcal{N} = 4$ SYM theory.

In the case of amplitudes the $N_c \to \infty$ limit coincides with topologically planar scalar integrals while for form factors this is not always true. One encounters contributions of the same order in N_c which contain planar and non-planar (from topological point of view) scalar integrals. This can be seen, for example, in studying of the Sudakov form factor $F_2 = \langle \phi_1^{AB} \phi_2^{AB} | T_{AB}^{(0)} | 0 \rangle$ [15] where the planar and non-planar diagrams contributing to the answer at the second order of perturbation theory are of the same order in N_c (as well as for form factors Z_n^{MHV} discussed above[9]). In the case of form factors for more general operators 1/2-BPS supermultiplet $Tr\Phi^n$, for example, the mixing between planar and non-planar scalar integrals of the same order in N_c happens at the n-th order. We will split the ratio $Z_3^{(m),MHV}/Z_3^{tree,MHV}$ into planar and non-planar parts in the sense of topology for scalar integrals and *will discuss the planar part only.*

The scalar integrals encountering in loop computations of the form factors of the half-BPS operators are not in general pseudo dual conformal invariant (see [15, 14]), so one may think that there are no restrictions for the basis of scalar integrals apart from the requirement of the UV finiteness. However, the computations based on the $\mathcal{N} = 1$ coordinate superspace suggest [15] that the basis of scalar integrals for the form factors of the half-BPS operators can be obtained from that of the MHV amplitudes applying the reduction procedure discussed in [15] which is based on the calculation of triangle and box-type ladder diagrams in [31]. This gives us the conjectured basis of scalar integrals for the MHV form factor with n legs at one and two loops. The coefficients of these integrals are fixed by $D = 4$ two-particle iterated cuts, much in the spirit of the five- and six-point two-loop computations for the MHV amplitudes.

Let us consider the form factor of the form $F_n = \langle \Omega_n | \mathcal{O}_{AB}^{(m)} | 0 \rangle$, where $n \geq m$ and

[8]In addition to dual pseudoconformal integrals, if dimensional reduction is used, there are contributions from the integrals over the ϵ-part of the loop momenta. Such integrals are not dual conformal; however, for the MHV sector they likely cancel among themselves if one considers the BDS exponent [29].

[9]We would like to thank G. Yang for pointing out this to us.

introduce the notion of the number of "interacting fields" V for the form factor

$$V = m_{in} + m_{fin}, \tag{3.70}$$

as the number of fields participating in the Wick contractions from the initial operator $\mathcal{O}_{AB}^{(m)}$ and from the interaction Lagrangian, m_{in}, plus the number of fields participating in the Wick contractions from the external super state $\langle\Omega_n|$ and the interaction Lagrangian, m_{fin} .

For example, the form factor $F_3 = \langle\phi_1^{AB}\phi_2^{AB}g_3^+|\mathcal{O}_{AB}^{(2)}|0\rangle$ has $V = 2+3$, $m_{in} = 2, m_{fin} = 3$ in any order of PT, and the form factor $F_n = \langle\phi_1^{AB}\ldots\phi_n^{AB}|\mathcal{O}_{AB}^{(n)}|0\rangle$ in the first order of PT $V = 2+2$, and at second order of PT V can take two possible values $V = 2+2$ and $V = 3+3$. This can be easily seen from examples of $\mathcal{N} = 1$ coordinate superspace Feynman diagrams for the $F_3 = \langle\phi_1^{AB}\phi_2^{AB}\phi_3^{AB}|\mathcal{O}_{AB}^{(3)}|0\rangle$ form factor from [15] presented in Fig. 1. In fact, the definition of V is needed to take into account the factorized contribution when the number of particles in the external state $|\Omega_n\rangle$ is equal to the number of fields in the operator $\mathcal{O}^{(n)}$.

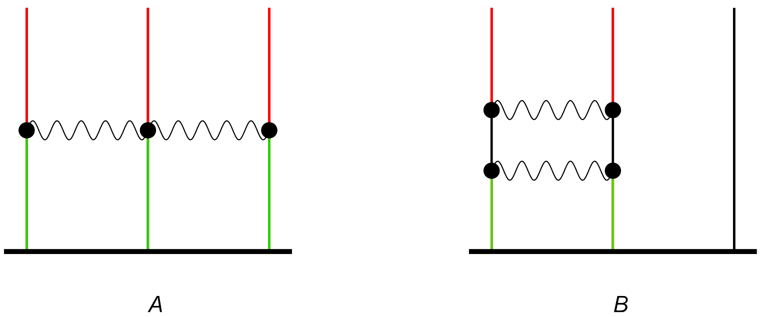

Figure 1: Some coordinate superspace Feynman diagrams for the $F_3 = \langle\phi_1^{AB}\phi_2^{AB}\phi_3^{AB}|\mathcal{O}_{AB}^{(3)}|0\rangle$ form factor in two loop order. The number of red lines is equal to m_{in}, the number of green lines equal to m_{fin}. The straight lines correspond to the chiral propagators $\langle\bar{\Phi}_I^a\Phi_J^b\rangle$, the wavy lines correspond to the vector propagator $\langle V^aV^b\rangle$ of $\mathcal{N} = 1$ superfields [15]. The lower bold line represents the insertion of the corresponding operator $\mathcal{O}_{AB}^{(3)}$.

Conjecture. The form factors $F_n = \langle\Omega_n|\mathcal{O}_{AB}^{(m)}|0\rangle$ in the l'th order of PT are given by scalar integrals which are obtained by reduction procedure from the scalar pseudoconformal integrals appearing at the l-loop in the V-point MHV amplitudes. This procedure involves the shrinking of $m_{in} - 1$ propagators which connect m_{in} external legs standing in a row for the corresponding amplitude. If the m_{in} neighboring momenta are attached to the same vertex, then there are no contractions. The shrinking of a propagator to a point can also be understood in dual variables as taking the $m_{in} - 1$ external legs (more accurately points) to infinity [15]. The integrals containing the UV divergent subgraphs

(bubbles) are not taken into account. Heuristically, this rule can be understood as the consequence of the appearance of a new type of the "effective vertex" in the unitarity cuts - the form factor, which glues several momenta together.

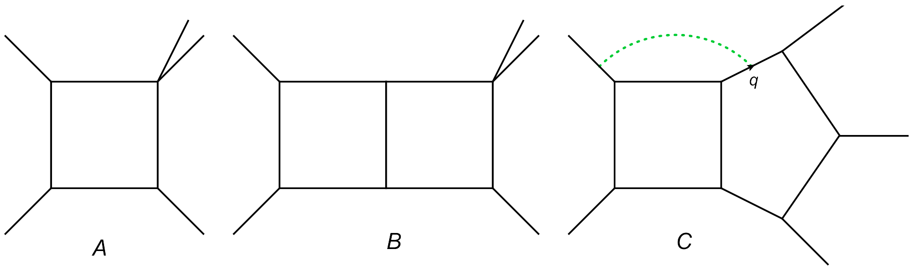

Figure 2: The basis of scalar integrals through $O(\epsilon)$ at one (A) and two (B,C) loops for the even part of the five-point MHV amplitude. Green ark corresponds to the presence of the numerator.

As an example of the above mentioned procedure one can apply it to find the basis of scalar integrals for one- and two-loop three-point super form factor from the corresponding one- and two-loop five-point MHV amplitude. At one loop one can choose as the basis for the integrals the combination of boxes (see Fig. 2 A), and at two loops - the combination of diagrams shown in Fig. 2 B and C. The result of the application of the reduction procedure is presented in Fig. 3 (we consider only the even part since the odd part gives no contribution for three-point kinematics of the form factor). Hence, one expects to obtain the answer for three-point MHV form factor in terms of the following set of scalar integrals:

$$\text{1-loop: } \mathbf{G}_1, \mathbf{G}_2,$$

$$\text{2-loops: } \mathbf{G}_3, \mathbf{G}_3[Num.], \mathbf{G}_4, \mathbf{G}_5.$$

All these integrals can be captured by two particle iterated cuts. Note also that the reduction procedure, at least in the current form, is "blind" to the external momenta configuration, and, in principle, one has to consider all different combinations. However, it is natural to assume that the off-shell momenta of the operator should be attached to the contracted propagator. We use the pattern which is represented in Fig. 4.

4 n-point MHV super form factor at one loop

Here we will reproduce the results obtained in [14] for the three-point MHV form factors using "superstate - super form factors", and briefly consider the case for a general n-point MHV super form factor. To obtain the full answer for the one loop MHV super form factor $Z_3^{(1),MHV}$, one has to consider the following cuts (see Fig. 5). Since the operator

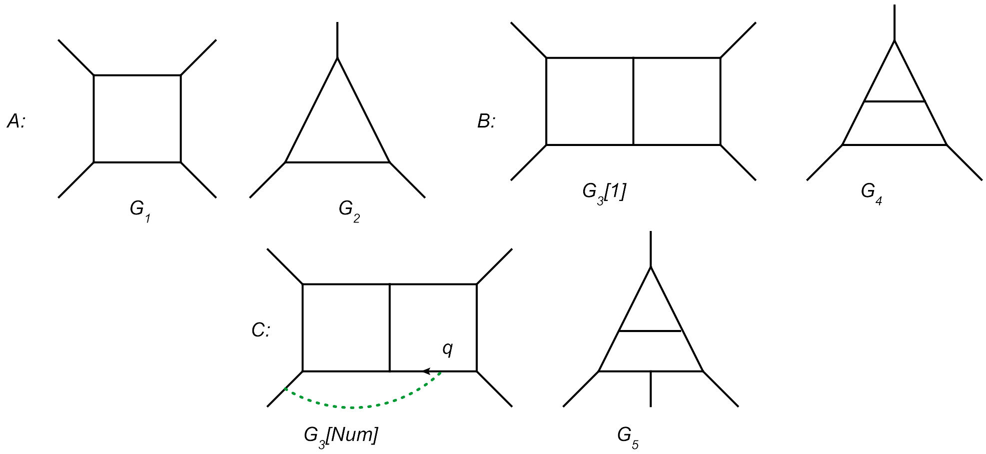

Figure 3: Scalar integrals obtained by the reduction procedure. We conjecture that they form a basis of scalar integrals for the $n = 3$-point MHV form factor at one (A) and two (B,C) loops.

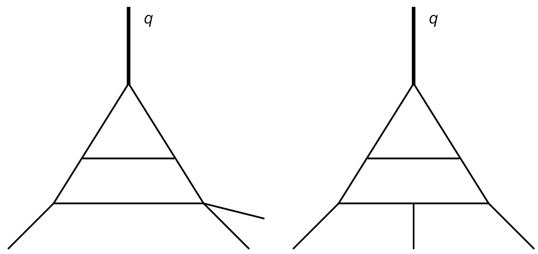

Figure 4: The configuration of external momenta for $\mathbf{G}_5$ and $\mathbf{G}_4$ scalar integrals.

is color singlet and the external state is color-ordered, all possible cyclic permutations of external momenta should be considered [14]. Let us consider first the cut A shown in Fig. 5

$$disc_{q^2}[Z_3^{(1),MHV}] = \int dSLIPS_2^{l_1 l_2} Z^{tree,MHV}(-l_1,-l_2)\mathcal{A}_5^{tree,MHV}(1,2,3,l_1,l_2) + \text{perm}, \tag{4.71}$$

where

$$q = l_1 + l_2 = p_1 + p_2 + p_3,$$

and perm means the terms with the exchanged momenta $(1,3)$ by $(2,1)$ and $(3,2)$. Performing the supersummation one gets

$$disc_{q^2}[Z_3^{(1),MHV}] = \lambda Z_3^{tree,MHV} \int dLIPS_2^{l_1 l_2} \frac{\langle l_1 l_2\rangle\langle 13\rangle}{\langle 1 l_1\rangle\langle 3 l_2\rangle} + \text{perm}. \tag{4.72}$$

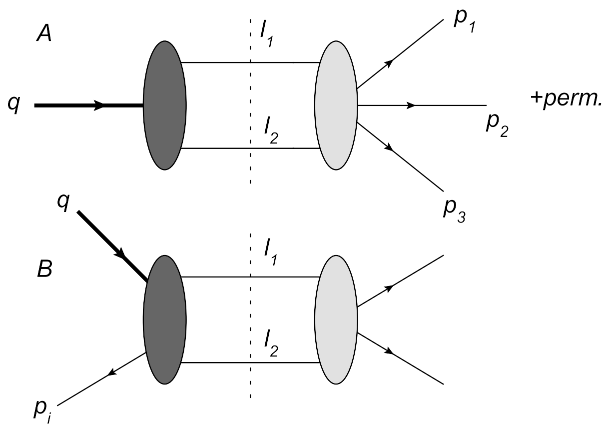

Figure 5: All two particle iterated cuts for the $n = 3$ point MHV form factor at one loop. Dark grey vertex corresponds to the MHV tree form factor, light grey vertex corresponds to the MHV tree amplitude.

The integrand is identical to the integrand obtained in [14]. Replacing $dLIPS_2^{l_1 l_2} \to d^D l_1/(2\pi)^D l_1^2 l_2^2$, where l_1 is an unrestricted loop momentum, we rearrange the integrand as

$$\frac{\langle l_1 l_2\rangle\langle 13\rangle}{\langle 3 l_1\rangle\langle l_2 1\rangle} = \frac{C_1}{2(l_2 1)} + \frac{C_3}{2(l_1 3)} + \frac{D_{13}}{2(l_1 3)2(l_2 1)}, \tag{4.73}$$

where the notation $p_i \equiv i$, $(p_i + p_j)^2 \equiv 2(ij)$ is used and also

$$C_i = (iq), D_{ij} = 2(ik)(kj), i \neq k, j \neq k. \tag{4.74}$$

We see the part of $Z_3^{(1),MHV}$ has a discontinuity in the q^2 channel

$$\begin{aligned} Z_3^{(1),MHV}|_{q^2} &= \lambda Z_3^{tree,MHV} M_3^{(1)}, \\ M_3^{(1)} &= \Big[D_{13}\mathbf{G}_1(1,2,3|(-q)^2) + C_1\mathbf{G}_2(1|(-q)^2, s_{23}) \\ &\quad + C_3\mathbf{G}_2(3|(-q)^2, s_{12})\Big] + \text{perm}, \end{aligned} \tag{4.75}$$

where perm means the permutation over the external legs which leads to the doubling of the triangles $\mathbf{G}_2$ in the above expression. The following notation for the integrals was used

$$\mathbf{G}_i[Num.](\text{massless legs}|(\text{massive legs})^2).$$

The case with the cut B from Fig. 5 does not bring any new information [14]. Nevertheless, let us consider this cut, since it introduces some features which will be encountered in the two-loop calculation. Let us choose $i = 3$

$$disc_{(q-3)^2}[Z_3^{(1),MHV}] = \int dSLIPS_2^{l_1 l_2} Z_2^{tree,MHV}(l_2, l_1, 3)\mathcal{A}_4^{tree,MHV}(1, 2, -l_1, -l_2), \tag{4.76}$$

where

$$q - p_3 = l_1 + l_2 = p_1 + p_2.$$

Performing the supersummation and replacing $dLIPS_2^{l_1 l_2} \to d^D l_1/(2\pi)^D l_1^2 l_2^2$ we can write the part of $Z_3^{(1),MHV}$ which has a discontinuity in the $(q-3)^2$ channel as

$$Z_3^{(1),MHV}|_{(q-3)^2} = \lambda Z_3^{tree,MHV} \int \frac{d^D l_1}{(2\pi)^D l_1^2 l_2^2} \hat{R}(3,3), \tag{4.77}$$

here $\hat{R}(a,b)$ is a universal one-loop function [7] defined as

$$\hat{R}(a,b) = R(b,a+1) + R(b-1,a) - R(a,b) - R(b-1,a+1), \tag{4.78}$$

where

$$\begin{aligned} R(b,a) &= 1 + \frac{C_b}{2(bl_1)} + \frac{C_a}{2(al_2)} + \frac{D_{ba}}{2(bl_1)2(al_2)}, \\ C_a &= (aP), D_{ab} = 2(aP)(bP) - P^2(ab). \end{aligned} \tag{4.79}$$

In our case $P = q - 3$ and it gives

$$\hat{R}(3,3) = R(3,1) + R(2,3) - R(3,3) - R(2,1),$$

where $R(3,3) = 1$ [14] and $D_{12} = 0$.

So $\hat{R}(3,3)$ takes the form

$$\hat{R}(3,3) = \frac{C_3}{2(3l_1)} + \frac{C_3}{2(3l_2)} + \frac{D_{31}}{4(3l_1)(1l_2)} + \frac{D_{23}}{4(2l_1)(3l_2)}. \tag{4.80}$$

Using this we can arrange $Z_3^{(1),MHV}|_{(q-3)^2}$ as (we are dropping the common $\lambda Z_3^{tree,MHV}$ prefactor)

$$D_{13}\mathbf{G}_1(1,2,3|(-q)^2) + D_{23}\mathbf{G}_1(2,1,3|(-q)^2) + 2C_3\mathbf{G}_2(3|(-q)^2, s_{12}). \tag{4.81}$$

All these integrals, as expected, are contained in (4.75). Similar cuts in the $(q-2)^2$- and $(q-1)^2$- channels should be considered in the same way, so the full answer for $Z_3^{(1),MHV}$ is given by the following expression

$$\begin{aligned} Z_3^{(1),MHV} &= \lambda Z_3^{tree,MHV} M_3^{(1)}, \qquad (4.82) \\ M_3^{(1)} &= D_{13}\mathbf{G}_1(1,2,3|(-q)^2) + D_{12}\mathbf{G}_1(1,3,2|(-q)^2) + D_{23}\mathbf{G}_1(2,1,3|(-q)^2) \\ &+ 2C_1\mathbf{G}_2(1|(-q)^2, s_{23}) + 2C_2\mathbf{G}_2(2|(-q)^2, s_{13}) + 2C_3\mathbf{G}_2(3|(-q)^2, s_{12}). \end{aligned}$$

The case with n legs is treated in a similar way. For the cuts in q^2- and $(q-i)^2$-channels one gets similar contributions as in [14]. Apart from the previous cuts there is

also another new kinematic channel in $s_{a+1,b-1} = (p_{a+1} + \ldots + p_{b-1})^2, a, b = 1, \ldots, n; a \neq b$. The result in this channel is

$$Z_n^{(1),MHV}|_{s_{a+1,b-1}} = \lambda Z_3^{tree,MHV} \int \frac{d^D l_1}{(2\pi)^D l_1^2 l_2^2} \hat{R}(a,b), \tag{4.83}$$

which coincides with the results of [14] and gives all the possible combination of two mass-easy and one mass-easy boxes in addition to triangles in the q^2 and $(q-i)^2$ channels.

Let us discuss several soft limits. First we consider $q \to 0$ for $n \geq 4$ (in the cases when $n = 2, 3$ the kinematics becomes degenerate). At the tree-level the connection is given by (2.65), so it is interesting to verify whether such a relation holds at the one-loop level. All the contributions in the q^2-channel and $(q-i)^2$-channel vanish in this limit, because all the coefficients such as C_i and D_{ij} contain at least a linear dependence on q. The remaining contributions coming from the $s_{a+1,b-1}, a \neq b$ channels reproduce exactly the one-loop n-point MHV amplitude [7]

$$Z_n^{(1),MHV}|_{s_{a+1,b-1}}^{q=0} = \mathcal{A}_n^{(1),MHV}|_{s_{a+1,b-1}} = \lambda \mathcal{A}_n^{tree,MHV} \int \frac{d^D l_1}{(2\pi)^D l_1^2 l_2^2} \hat{R}(a,b). \tag{4.84}$$

Let us consider also the three-point form factor in the limit where the momentum of the third external particle becomes soft. We would expect that in this limit the ratio $Z_3^{(1),MHV}/Z_3^{tree,MHV}$ would give us the one-loop planar part of the $M_2^{(1),planar}$ form factor – the one-loop Sudakov form factor [11, 15]. Indeed, one can see that in this limit all D_{ij} in (4.75) vanish and $\mathbf{G}_2$ integrals give us just the necessary combination

$$M_2^{(1),planar} = 2 s_{12} \mathbf{G}_2(1, 2|s_{12}). \tag{4.85}$$

We would expect also that similar behaviour holds also at the higher-loop level. For more details on the soft and collinear limits see Appendix B.

5 Ansatz for three-point MHV super form factor at two loops

We consider here the $D = 4$ two-particle iterated cuts for three-point super form factors at two loops based on two-particle iterated cuts, and suggest the ansatz for the three-point super form factors at two loops based on the assumed basis of scalar integrals (which is obtained by reduction from the basis of the scalar pseudoconformal integrals for the amplitudes). The needed unitary cuts for the three-point form factor are shown in Fig. 6.

Let us consider the cut A first. Using the momentum conservation relation

$$q = l_1 + l_2 = m_1 + m_2 = p_1 + p_2 + p_3,$$

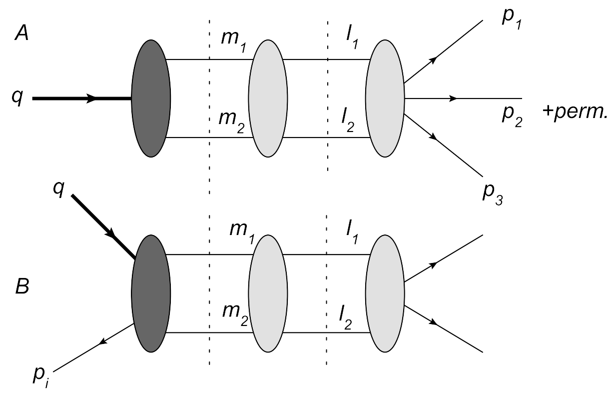

Figure 6: All two-particle iterated cuts for the three-point MHV form factor at two loops. Dark grey vertex corresponds to he MHV tree form factor, while the light grey vertex to the MHV tree amplitude.

performing the supersummation and replacing $dLIPS_4^{l_1 l_2 m_1 m_2} \to d^D l_1 d^D m_1/(2\pi)^{2D} l_1^2 l_2^2 m_1^2 m_2^2$ where l_1, m_1 are unrestricted loop momenta one writes the part of $Z_3^{(2),MHV}$ which has a double discontinuity (i.e. discontinuity of discontinuity) in the q^2-channel as

$$\lambda^2 Z_3^{tree,MHV} \int \frac{d^D l_1 d^D m_1}{(2\pi)^D (2\pi)^D} \frac{1}{l_1^2 l_2^2 m_1^2 m_2^2} \frac{\langle 31 \rangle \langle l_1 l_2 \rangle^2 \langle m_1 m_2 \rangle}{\langle 1 l_1 \rangle \langle 3 l_2 \rangle \langle l_1 m_1 \rangle \langle l_2 m_2 \rangle} + \text{perm}, \tag{5.86}$$

which is combined into (we are dropping the common $\lambda^2 Z_3^{tree,MHV}$ prefactor)

$$\begin{aligned} q^2 \Big[& D_{13} \mathbf{G}_5(1,2,3|(-q)^2) + 2C_1 \mathbf{G}_4(1|(-q)^2, s_{23}) \\ & + 2C_3 \mathbf{G}_4(3|(-q)^2, s_{12}) \Big] + \text{perm}, \end{aligned} \tag{5.87}$$

from which we see that all the conjectured integrals except for some double boxes were captured by the cut A.

Let us consider now the cut B and take $i = 3$ (for any other i the procedure is the same). Taking into account the momentum conservation laws which are

$$q - p_3 = l_1 + l_2 = m_1 + m_2 = p_1 + p_2,$$

and performing the supersummation one can rewrite the part of $Z_3^{(2),MHV}$ which has the double discontinuity in the $(q-3)^2$-channel as

$$\lambda^2 Z_3^{tree,MHV} (12) \int \frac{d^D m_1}{(2\pi)^D m_1^2 m_2^2} \text{Box}(2, m_1)\ \hat{R}(3,3), \tag{5.88}$$

where

$$\text{Box}(x,y) = \int \frac{d^D l_1}{(2\pi)^D l_1^2 l_2^2} \frac{1}{(x l_2)(y l_2)}.$$

Using the properties of $\hat{R}(3,3)$ discussed in the previous section one gets (again the common $(12)\lambda^2 Z_3^{tree,MHV}$ factor is dropped)

$$\mathbf{G}_3[C_3(2+m_1)^2 + D_{13}](1,2,3|(-q)^2) + \mathbf{G}_3[C_3(1+m_1)^2 + D_{32}](1,2,3|(-q)^2),$$

thus obtaining the expected double boxes.

The cuts in the $(q-1)^2$- and $(q-2)^2$- channels give us the same boxes with the numerators

$$(23)C_1(3+m_1)^2 + (23)D_{21},\ (23)C_1(2+m_1)^2 + (23)D_{13},$$
$$(13)C_2(3+m_1)^2 + (13)D_{12},\ (13)C_2(1+m_1)^2 + (13)D_{23}.$$

Combining all contributions together we can write the following ansatz for the three-point MHV two-loop form factor

$$Z_3^{(2),MHV} = \lambda^2 Z_3^{tree,MHV} M_3^{(2)},$$

with

$$\begin{aligned} M_3^{(2)} = \Big[& q^2 D_{13}\mathbf{G}_5(1,2,3|(-q)^2) + 2q^2 C_1 \mathbf{G}_4(1|(-q)^2, s_{23}) \\ & + 2q^2 C_3 \mathbf{G}_4(3|(-q)^2, s_{12}) + s_{12}\mathbf{G}_3[C_3(2+m_1)^2 + D_{13}](1,2,3|(-q)^2) \\ & + s_{12}\mathbf{G}_3[C_3(1+m_1)^2 + D_{32}](1,2,3|(-q)^2)\Big] + \text{perm.} \end{aligned} \tag{5.89}$$

We see that this expression in the limit when momenta for the third external leg goes to zero gives the expected expression [11, 15]

$$\frac{Z_3^{(2),MHV}}{Z_3^{tree,MHV}}|_{3\to 0} = M_2^{(2),planar} = 4s_{12}^2 \mathbf{G}_4(1,2|s_{12}). \tag{5.90}$$

This ansatz for the three-point MHV form factor in the second order of PT is based on the conjecture that the basis of scalar integrals for the form factors is obtained from the basis of scalar integrals for particular (MHV) amplitudes, using the reduction procedure first introduced in [15] and studied here for the purposes of our calculations. It still needs verification concerning the factorization properties, collinear and soft limits, as well as the computation none planar (in the sense of the topology of the scalar integrals) contribution. We are going to address these questions in upcoming publications.

6 Discussion

In this paper, the systematic study of form factors in the $\mathcal{N}=4$ SYM theory is performed. Initially, they were studied long time ago in [11], [12] and revived recently first in the strong coupling [13] and then in the weak coupling regime in [15], where the $\mathcal{N}=1$ superspace calculation was used, and in [14], where the unitary-based technique for constructing the

one-loop form factors was applied. In this paper the manifestly $\mathcal{N}=4$ SUSY covariant answer is obtained for the form factor of the operator belonging to the stress-tensor energy supermultiplet, and the completely $\mathcal{N}=4$ SUSY covariant unitary-based technique for constructing the loop form factors is presented.

The central result of this paper is the formula for the MHV tree level supersymmetric form factor (2.62) for the operators belonging to the stress-tensor energy supermultiplet (its chiral truncation) and its connection to the superamplitude given by (2.69). This hypothesis links the super form factor with super momentum equal to zero and the derivative of the superamplitude. The latter conjecture has been verified at the tree- and one-loop level for any MHV n-point ($n \geq 4$) form factor involving the stress-tensor energy supermultiplet. Apart from this the collinear and soft limits of form factors were considered which are very similar to the corresponding limits for the amplitudes. In the case of form factors, there are two different soft limits: the first one is applied to one of the external on-shell legs bringing the form factor with n external legs to the form factor with $n-1$ legs. The other one is the soft limit applied to the momentum of the initial operator which brings to the amplitude.

The two-loop result for the three-point super form factor is constructed based on two-particle iterative cuts and the hypothesis that the basis of scalar integrals is known. It was known for a long time that the two-point form factor in the $\mathcal{N}=4$ SYM theory has a very simple structure [11]: the anomalous cusp and collinear dimensions determines the divergent part while the finite part is trivial (constant, and is conjectured to be completely determined by the one loop result in some renormalization scheme [11]) and exponentiates. This is very similar to the behavior of the four- and five-point MHV amplitudes where the finite part is also simple and exponentiates. The latter can be understood as a consequence of the dual conformal invariance. It is natural to consider the two-point form factor as an analog of the four-point amplitude [32] and three-point form factor corresponds to the five-point one [30]. Then it is interesting to verify if the two-loop answer for the three-point form factor obtained here has a simple structure and exponentiate. This requires the calculation of the basic scalar integrals. If it is really the case it would be the indication that the form factors are also governed by some hidden symmetry.

The basis of scalar integrals appearing in form factor calculation deserves a separate comment. This basis is not arbitrary but can be obtained by the reduction procedure from the pseudoconformal integrals which appear in the amplitude calculations. This procedure was observed earlier in [15] where a completely different approach was used for studying the form factors, namely, the $\mathcal{N}=1$ superspace technique. The main idea of this reduction procedure is the following: to get the scalar integrals for form factors, one should shrink some of the propagators in scalar pseudoconformal integrals forming the basis of integrals for the amplitudes.

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Note added: while finishing writing the paper we became aware of the paper which is closely connected to the subject studied here and has an overlap in deriving the super form factor [33].

A $\mathcal{N}=4$ harmonic superspace

We discuss here the reformulation of (2.62) in the $\mathcal{N}=4$ harmonic superspace. Our discussion is based mostly on section 3 of [24]. The $\mathcal{N}=4$ harmonic superspace is obtained by adding additional bosonic coordinates (harmonic variables) to the $\mathcal{N}=4$ coordinate superspace or on-shell momentum superspace. These additional bosonic coordinates parameterize the coset

$$\frac{SU(4)}{SU(2)\times SU(2)\times U(1)}$$

and carry the $SU(4)$ index A, two copies of $SU(2)$ indices $a, \dot{a}$ and $U(1)$ charge $\pm$

$$(u^{+a}_{A},\ u^{-\dot{a}}_{A}).$$

Using these variables one presents all the Grassmannian objects with $SU(4)_R$ indices. For example, for Grassmannian coordinates in the original $\mathcal{N}=4$ coordinate superspace

$$\theta^{+a}_{\alpha}=u^{+a}_{A}\theta^{A}_{\alpha},\ \theta^{-\dot{a}}_{\alpha}=u^{-\dot{a}}_{A}\theta^{A}_{\alpha}, \tag{A.91}$$

and in the opposite direction

$$\theta^{A}_{\alpha}=\theta^{+a}_{\alpha}u^{A}_{+a}+\theta^{-\dot{a}}_{\alpha}\bar{u}^{A}_{+a}. \tag{A.92}$$

The same can be done with supercharges, projection operators from (2.28) etc.. Note that harmonic variable projection leaves helicity properties of the objects unmodified. Also, similar projections can be performed for Grassmannian coordinates η^A and supercharges $q^A_\alpha,\ \bar{q}_{\dot{\alpha}A}$ of on-shell momentum superspace.

So the $\mathcal{N}=4$ harmonic superspace is parameterized with the following set of coordinates

$$\begin{aligned} \mathcal{N}=4 \text{ harmonic superspace} \ &= \ \{x^{\alpha\dot{\alpha}},\ \theta^{+a}_{\alpha},\theta^{-\dot{a}}_{\alpha},\ \bar{\theta}^{+a}_{\dot{\alpha}},\bar{\theta}^{-\dot{a}}_{\dot{\alpha}},u\} \\ \text{or} \ &\quad\ \{\lambda_\alpha,\tilde{\lambda}_{\dot{\alpha}},\ \eta^{+a},\eta^{-\dot{a}},\ u\}. \end{aligned} \tag{A.93}$$

Using u harmonic variables one can project the W^{AB} superfield as

$$W^{AB} \to W^{AB} u^{+a}_{A} u^{+b}_{B} = \epsilon^{ab} W^{++},$$

where ϵ^{ab} is an $SU(2)$ totally antisymmetric tensor and the Grassmannian analyticity conditions (2.13) such that[10]

$$D^{\alpha}_{-\dot{a}} W^{++} = 0,\ \bar{D}^{\dot{\alpha}}_{+a} W^{++} = 0.$$

Thus, the superfield W^{++} contains the dependence on half of the Grassmannian variables θ's and $\bar{\theta}$'s.

$$W^{++} = W^{++}(x,\ \theta^{+a}_{\alpha}, \bar{\theta}^{-\dot{a}}_{\dot{\alpha}}, u),$$

Performing the expansion of W^{++} in u *all* the projections like (2.12) in $SU(4)_R$ covariant fashion can be obtained. This is the main purpose of introduction of the harmonic superspace. The component expansion of W^{++} in θ's and $\bar{\theta}$'s can be found in [24]. The lowest component of the W^{++} expansion is

$$W^{++}(x, 0, 0, u) = \phi^{++},\ \phi^{++} = \frac{1}{2} \epsilon_{ab} u^{+a}_{A} u^{+b}_{B} \phi^{AB},$$

where according to [24]

$$Q^{-\dot{a}}_{\alpha} \phi^{++} = 0. \tag{A.94}$$

Using this condition we can write in analogy with (2.41) that the form factor

$$F_n(\{\lambda, \tilde{\lambda}, \eta\}, q, u) = \langle \Omega_n | Tr(\phi^{++}\phi^{++}) | 0 \rangle, \tag{A.95}$$

satisfies the following condition:

$$q^{-\dot{a}}_{\alpha} F_n(\{\lambda, \tilde{\lambda}, \eta\}, q, u) = 0,$$

where $q^{-\dot{a}}_{\alpha}$ is the projected supercharge in the on shell momentum superspace representation $q^{-\dot{a}}_{\alpha} = u^{-\dot{a}}_{A} q^{A}_{\alpha}$. We see that

$$F_n \sim \delta^{-4}(q^{\dot{a}}_{\alpha})(\ldots),$$

where δ^{-4} is the Grassmannian delta function; $\delta^{\pm 4}$ are defined as

$$\delta^{\pm 4}(q^{a/\dot{a}}_{\alpha}) = \sum_{i,j=1}^{n} \prod_{a/\dot{a}, b/\dot{b}=1}^{2} \langle ij \rangle \eta^{\pm a/\dot{a}}_{i} \eta^{\pm b/\dot{b}}_{j}. \tag{A.96}$$

[10] Strictly speaking this is true only in the free theory ($g = 0$), in the interacting theory one has to replace $D^{A}_{\alpha}, \bar{D}^{A}_{\dot{\alpha}}$ by their gauge covariant analogs, which contain superconnection, but the final result is the same [24].

Since the u projections do not change the helicity structure of Grassmannian variables, using the previous arguments we can write at the tree level

$$F_n^{tree,MHV}(\{\lambda,\tilde{\lambda},\eta\},q,u) \sim \frac{\delta^{-4}(q^a_\alpha)}{\langle 12\rangle\ldots\langle n1\rangle}.$$

This expression is an $SU(4)_R$ invariant analog of (2.49), which contains all form factors of the form $\langle\Omega_n|T^{AB}_{(0)}|0\rangle$.

The superfield W^{++} can also be used to combine all the components of the chiral sector of T^{AB}. Letting $\bar{\theta}$'s in W^{++} to 0 the superfield $\mathcal{T}$ [24]

$$\mathcal{T}(x,\ \theta^{+a}_\alpha,u) = Tr(W^{++}W^{++})(x,\ \theta^{+a}_\alpha,u),$$

contains all projections like $\mathcal{T}^{ab}$. We can write $\mathcal{T}$ as

$$\mathcal{T}(x,\ \theta^{+a}_\alpha,u) = e^{Q^{+a}_\alpha\theta^\alpha_{+a}}\mathcal{T}(x,0,u),$$

so that for "super state -super form factor" we have

$$\mathcal{F}_n(\{\lambda,\tilde{\lambda},\eta\},q,u,\theta^{+a}_\alpha) = \langle\Omega_n|\mathcal{T}(\theta^{+a}_\alpha)|0\rangle = e^{q^{+a}_\alpha\theta^\alpha_{+a}}F_n(\{\lambda,\tilde{\lambda},\eta\},q,u). \tag{A.97}$$

One can define an analog of $\hat{T}$ transformation for θ^α_{+a}, so that for the MHV part of $\mathcal{F}_n$ at tree level one can write:

$$\hat{T}[\mathcal{F}_n^{tree,MHV}] \sim \frac{\delta^{+4}(q^a_\alpha+\lambda^{'}_\alpha\eta^{'a}+\lambda^{''}_\alpha\eta^{''a})\delta^{-4}(q^{\dot{a}}_\alpha)}{\langle 12\rangle\ldots\langle n1\rangle}. \tag{A.98}$$

This expression looks just like (2.62), but now both the Grassmannian delta functions δ^4 are $SU(4)_R$ covariant. One can write also the MHV part of a superamplitude in a similar manner. Projecting the condition of superamplitude invariance under q^A_α supersymmetry transformations we have

$$q^A_\alpha\mathcal{A}_n^{tree,MHV} = 0 \to (q^{+a}_\alpha + q^{-\dot{a}}_\alpha)\mathcal{A}_n^{tree,MHV} = 0,$$

and taking into account that the helicity properties of projected supercharges are not modified we get

$$\mathcal{A}_n^{tree,MHV} \sim \frac{\delta^{+4}(q^a_\alpha)\delta^{-4}(q^{\dot{a}}_\alpha)}{\langle 12\rangle\ldots\langle n1\rangle}. \tag{A.99}$$

Now both $\mathcal{A}_n^{tree,MHV}$ and $\hat{T}[\mathcal{F}_n^{tree,MHV}]$ are $SU(4)_R$ invariant and one can use them in unitarity based computations, where super summation should be performed separately for δ^{+4} and δ^{-4}.

B Collinear and soft limits

Here we gather together the results for collinear and soft limits for the form factors at the tree- and one-loop levels. In the limit when momenta i of external particle becomes soft at the tree level it is easy to see that similarly to the soft limit in the amplitudes [7] one has

$$Z_n^{tree,MHV} \stackrel{i\to 0}{\to} \mathrm{Soft}^{tree}(a,i,b) Z_{n-1}^{tree,MHV},\ n \geq 3,$$

where the $\mathrm{Soft}^{tree}(a,i,b)$ is the "eikonal" factor

$$\mathrm{Soft}(a,i,b) = \frac{\langle ab \rangle}{\langle ai \rangle \langle ib \rangle}, \tag{B.100}$$

with a and b being the momenta of color-ordered neighbors of the particle with momentum i.

At the one-loop level taking the soft limit in expressions (4.72), (4.77), and (4.83) one obtains

$$\frac{Z_n^{(1),MHV}}{Z_n^{tree,MHV}}|_{i\to 0} \sim M_{n-1}^{(1)},\ n \geq 4.$$

$Z_3^{(m),MHV}$ in general contains the planar and none planar scalar integrals. Taking the limit when the momentum of the i-th external leg goes to zero, one gets $Z_2^{(m),MHV}$ which is the Sudakov form factor. The color structure in this case is trivial; one has $Tr(T^a T^b) = \frac{1}{2}\delta^{ab}$ and the planar and non-planar diagrams are of the same order in N_c, as was explained earlier. One would expect that relation between $Z_3^{(m),MHV}$ and the part of $Z_2^{(m),MHV}$, which contains the planar and none planar diagrams will hold separately

$$\frac{Z_3^{(m),MHV,planar/none\ planar}}{Z_3^{tree,MHV}}|_{3\to 0} \sim M_2^{(m),planar/none\ planar}, \tag{B.101}$$

at one and two loops.

Taking the soft limit when the (super)momenta of an operator becomes soft one derives

$$Z_n^{tree/(1),MHV}(\{\lambda,\tilde{\lambda},\eta\},0,\{0\}) = \mathcal{A}_n^{tree/(1),MHV}(\lambda,\tilde{\lambda},\eta). \tag{B.102}$$

Let us now consider the collinear limits. For a particular component expression

$$F_5^{tree,MHV} = \langle \phi_1^{AB} \phi_2^{AB} g_3^+ g_4^+ g_5^+ | T_{AB}^{(0)}(q) | 0 \rangle$$

one has

$$\begin{aligned}
&F_5^{tree,MHV} \stackrel{\phi_1^{AB}||\phi_2^{AB}}{\to} 0,\\
&F_5^{tree,MHV} \stackrel{\phi_2^{AB}||g_3^+}{\to} \mathrm{Split}_\phi(\phi,g^+) F_4^{tree,MHV},\\
&F_5^{tree,MHV} \stackrel{g_3^+||g_4^+}{\to} \mathrm{Split}_-(g^+,g^+) F_4^{tree,MHV},\\
&F_5^{tree,MHV} \stackrel{g_5^+||\phi_1^{AB}}{\to} \mathrm{Split}_\phi(g^+,\phi) F_4^{tree,MHV},
\end{aligned} \tag{B.103}$$

where the splitting functions are defined by the following expressions:

$$\begin{aligned}
\mathrm{Split}_\phi(\phi, g^+) &= \frac{1}{\langle ij\rangle}\sqrt{\frac{z}{1-z}},\\
\mathrm{Split}_\phi(g^+, \phi) &= \frac{1}{\langle ij\rangle}\sqrt{\frac{1-z}{z}},\\
\mathrm{Split}_-(g^+, g^+) &= \frac{1}{\langle ij\rangle}\frac{1}{\sqrt{z(1-z)}},
\end{aligned} \tag{B.104}$$

where i and j correspond to collinear momenta.

Thus, similar to the collinear limit in the case of the MHV amplitudes the following relation holds

$$F_n^{tree,MHV}(\ldots, p_a^{\lambda_i}, p_b^{\lambda_{i+1}}, \ldots) \stackrel{i||i+1}{\to} \sum_{\lambda,c} \mathrm{Split}_{-\lambda}(a^{\lambda_i}, b^{\lambda_{i+1}}, z) F_{n-1}^{tree,MHV}(\ldots, p_c^{\lambda}, \ldots).$$

References

[1] N. Beisert, *On Yangian Symmetry in Planar $\mathcal{N} = 4$ SYM*, arXiv:1004.5423v2 [hep-th].

[2] T. Bargheer, N. Beisert, W. Galleas, F. Loebbert, T. McLoughlin, *Exacting $\mathcal{N} = 4$ Superconformal Symmetry*, JHEP **0911** (2009) 056, arXiv:0905.3738v3 [hep-th]; N. Beisert, J. Henn, T. McLoughlin, J. Plefka, *One-Loop Superconformal and Yangian Symmetries of Scattering Amplitudes in $\mathcal{N} = 4$ Super Yang-Mills*, JHEP **1004** (2010) 085, arXiv:1002.1733v2 [hep-th].

[3] A. Gorsky, *Amplitudes in the $\mathcal{N} = 4$ SYM from Quantum Geometry of the Momentum Space*, Phys. Rev. D **80** (2009) 125002, arXiv:0905.2058 [hep-th].

[4] L. F. Alday, J. Maldacena, A. Sever and P. Vieira, *Y-system for Scattering Amplitudes*, J. Phys. A **43** (2010) 485401, arXiv:1002.2459 [hep-th].

[5] F. Cachazo, P. Svrcek and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, JHEP **0409** (2004) 006, arXiv:hep-th/0403047; A. Brandhuber, B. J. Spence, G. Travaglini, *One-loop gauge theory amplitudes in N=4 super Yang-Mills from MHV vertices*, Nucl. Phys. B **7 06** (2005) 150, arXiv:hep-th/0407214.

[6] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, Nucl. Phys. B **715** (2005) 499, arXiv:hep-th/0412308; R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, Phys. Rev. Lett. **94** (2005) 181602, arXiv:hep-th/0501052.

[7] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, *One-Loop n-Point Gauge Theory Amplitudes, Unitarity and Collinear Limits*, Nucl. Phys. B **425** (1994) 217,

arXiv:hep-ph/9403226; *Fusing gauge theory tree amplitudes into loop amplitudes*, Nucl. Phys. B **435** (1995) 59, arXiv:hep-ph/9409265.

[8] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $\mathcal{N}=4$ super-Yang-Mills*, Nucl. Phys. B **725** (2005) 275, arXiv:hep-th/0412103.

[9] J. M. Drummond, J. Henn, G. P. Korchemsky, E. Sokatchev,*Generalized unitarity for $\mathcal{N}=4$ super-amplitudes*, arXiv:0808.0491 [hep-th].

[10] M. Bianchi, H. Elvang, D. Z. Freedman, *Generating Tree Amplitudes in $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SG*, JHEP **0809** (2008) 063, arXiv:0805.0757 [hep-th].

[11] W. L. van Neerven, *Infrared Behavior Of On-Shell Form-Factors In a $\mathcal{N}=4$ Supersymmetric Yang-Mills Field Theory*, Z. Phys. C **30** (1986) 595.

[12] K.G. Selivanov, *On tree form-factors in (supersymmetric) Yang-Mills theory*, Commun. Math. Phys. **208** (2000) 671, hep-th/9809046.

[13] J. Maldacena and A. Zhiboedov, *Form factors at strong coupling via a Y-system*, JHEP **1011** (2010) 104, arXiv:1009.1139 [hep-th].

[14] A. Brandhuber, B. Spence, G. Travaglini and G. Yang, *Form Factors in $\mathcal{N}=4$ Super Yang-Mills and Periodic Wilson Loops*, JHEP **1101** (2011) 134, arXiv:1011.1899 [hep-th].

[15] L.V. Bork, D.I. Kazakov, G.S. Vartanov, *On form factors in $\mathcal{N}=4$ SYM*, JHEP **1102** (2011) 063, arXiv:1011.2440 [hep-th].

[16] S. Raju, *Recursion Relations for AdS/CFT Correlators*, Phys. Rev. **D83** (2011) 126002, arXiv:1102.4724 [hep-th].

[17] B. Eden, G. P. Korchemsky and E. Sokatchev, *From correlation functions to scattering amplitudes*, arXiv:1007.3246 [hep-th]; *More on the duality correlators/amplitudes*, arXiv:1009.2488 [hep-th].

[18] L. F. Alday, J. M. Maldacena, *Gluon scattering amplitudes at strong coupling*, JHEP **0706** (2007) 064, arXiv:0705.0303 [hep-th].

[19] J. M. Drummond, G. P. Korchemsky and E. Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops*, Nucl. Phys. B **795** (2008) 385, arXiv:0707.0243 [hep-th].

[20] A. Brandhuber, P. Heslop and G. Travaglini, *MHV Amplitudes in $\mathcal{N}=4$ Super Yang-Mills and Wilson Loops*, Nucl. Phys. B **794** (2008) 231, arXiv:0707.1153 [hep-th]. J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, *On planar gluon amplitudes/Wilson loops duality*, Nucl. Phys. B **795** (2008) 52, arXiv:0709.2368 [hep-th].

[21] Simon Caron-Huot, *Notes on the scattering amplitude / Wilson loop duality*, JHEP **1107** (2011) 058, arXiv:1010.1167 [hep-th].

[22] Z. Bern, J.J.M. Carrasco, H. Ita, H. Johansson, *On the Structure of Supersymmetric Sums in Multi-Loop Unitarity Cuts*, Phys. Rev. **D80** (2009) 065029 0903.5348 [hep-th].

[23] J.M. Drummond, J. Henn, G.P. Korchemsky, E. Sokatchev, *Dual superconformal symmetry of scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory*, Nucl. Phys. B **828** (2010) 317, arXiv:0807.1095 [hep-th].

[24] Burkhard Eden, Paul Heslop, Gregory P. Korchemsky, Emery Sokatchev, *The super-correlator/super-amplitude duality: Part I*, arXiv:1103.3714v1 [hep-th].

[25] Henriette Elvang, Yu-tin Huang, Cheng Peng, *On-shell superamplitudes in $\mathcal{N} < 4$ SYM*, arXiv:1102.4843v1 [hep-th].

[26] G.P. Korchemsky, E. Sokatchev, *Twistor transform of all tree amplitudes in $\mathcal{N} = 4$ SYM theory*, Nucl. Phys. **B829** (2010) 478, arXiv:0907.4107 [hep-th].

[27] A. Zhiboedov, unpublished notes, spring 2011.

[28] Z. Bern, M. Czakon, L.J. Dixon, D.A. Kosower and V.A. Smirnov, *The Four-Loop Planar Amplitude and Cusp Anomalous Dimension in Maximally Supersymmetric Yang-Mills Theory*, Phys. Rev. D **75** (2007) 085010, arXiv:hep-th/0610248;
F. Cachazo, M. Spradlin and A. Volovich, *Four-Loop Cusp Anomalous Dimension From Obstructions*, Phys. Rev. D **75** (2007) 105011, arXiv:hep-th/0612309.

[29] Z. Bern, Lance J. Dixon, D.A. Kosower, R. Roiban, M. Spradlin, C. Vergu, A. Volovich, *The Two-Loop Six-Gluon MHV Amplitude in Maximally Supersymmetric Yang-Mills Theory*, Phys.Rev.D78:045007,2008, arXiv:0803.1465 [hep-th].

[30] Cachazo, M. Spradlin and A. Volovich, *Iterative structure within the five-particle two-loop amplitude*, Phys. Rev. D **74** (2006) 045020, arXiv:hep-th/0602228;
Z. Bern, M. Czakon, D. A. Kosower, R.Roiban and V. A. Smirnov, *Two-Loop Iteration of Five-Point $\mathcal{N}$=4 Super-Yang-Mills Amplitudes*, Phys. Rev. Lett. **97** (2006) 181601, arXiv:hep-th/0604074.

[31] N. I. Usyukina and A. I. Davydychev, *An Approach to the evaluation of three and four point ladder diagrams*, Phys. Lett. B **298** (1993) 363.

[32] Z. Bern, L. J. Dixon and V. A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, Phys. Rev. D **72** (2005) 085001, arXiv:hep-th/0505205.

[33] A. Brandhuber, O. Gurdogan, R. Mooney, G. Travaglini and G. Yang, *Harmony of Super Form Factors*, arXiv:1107.5067 [hep-th].